On the Stochastic Burgers Equation and the Axiom of Choice

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Abstract

This article considers the stochastic Burgers equation

$$\begin{cases} \partial_t u^{(\epsilon)} = (\frac{\epsilon}{2} u^{(\epsilon)}_{xx} - \frac{1}{2}(u^{(\epsilon)2})_x)dt + \partial_t \zeta_x \\ u_0 \equiv 0 \end{cases}$$

where ζ is a spatially homogeneous Gaussian random field, 2π-periodic in the space variable, mean zero and with covariance $E_{\mathbf{Q}}\left\{\zeta(t,x)\zeta(s,y)\right\} = (s \wedge t)\sum_{n\geq 1} a_n^2 \cos(n(x-y)) = (s\wedge t)\Gamma(x-y)$ where Γ is 8 times differentiable. The main result is that

$$\lim_{\epsilon\to 0} \sup_{0\leq t\leq T} \sup_{x\in[0,2\pi]} \left| E_{\mathbf{Q}}\left\{u^{(\epsilon)2p}(t,x)\right\} - (-\Gamma''(0))^p \prod_{j=1}^{p}(2j-1)t^p \right| = 0.$$

for each positive integer p and each $T < +\infty$. This result is of interest following the work of E, Khanin, Mazel and Sinai, proving existence of invariant measure for the stochastic inviscid Burgers equation, where the hypotheses on the Gaussian random field include those of this article. The method of E, Khanin, Mazel and Sinai is to construct a solution to the Stochastic Inviscid Burgers equation using the minimising trajectories of the associated action functional. This construction relies on the fact that the minimiser exists, which depends on the relative weak compactness of the unit ball in L^2 (Tychonov compactness). Kelley proved that Tychonov compactness is equivalent to the Axiom of Choice. This article therefore demonstrates that the Axiom of Choice leads to contradictory results in mathematical analysis.

1 Summary and Notations

This article considers the stochastic Burgers equation

$$\begin{cases} \partial_t u^{(\epsilon)} = (\frac{\epsilon}{2} u^{(\epsilon)}_{xx} - \frac{1}{2}(u^{(\epsilon)2})_x)dt + \partial_t \zeta_x \\ u_0 \equiv 0 \end{cases} \tag{1}$$

where ∂_t denotes a stochastic differential with respect to the variable t, subscripts denote derivative with respect to the argument labelled by the subscripted variable and ζ is a space homogeneous, Gaussian, random field, satisfying the following hypotheses.

Hypothesis 1. • $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{0\leq t<+\infty}, \mathbf{Q})$ *denotes a filtered probability space, where* $(\beta^{jn})_{j=1,2;n\geq 1}$ *are standard independent Wiener processes with respect to* $\mathbf{Q}$*,* $\beta^{jn}(0) = 0$ *for each* $(j,n) \in \{1,2\}\times\mathbf{N}$ *and* $\mathcal{G}_t$ *is the* σ*-algebra generated by the increments*

$$\{(\beta^{jn}(v) - \beta^{jn}(u))_{0\leq u\leq v\leq t},\ (j,n)\in\{1,2\}\times\mathbf{N}\}.$$

- *The Gaussian random field ζ is defined as*

$$\zeta(t,x)=\sum_{n=1}^{\infty}a_n(\cos(nx)\beta^{1n}(t)+\sin(nx)\beta^{2n}(t))$$

where $(a_n)_{n\geq 1}$ are real numbers satisfying $\sum_{n=1}^{\infty}n^4|a_n|<+\infty$ and β^{jn} are independent standard Brownian motions such that $\beta^{jn}(0)=0$ with respect to $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\geq 0},\mathbf{Q})$.

Notation $E_{\mathbf{Q}}\{.\}$ denote the expectation operator with respect to the measure $\mathbf{Q}$.

Γ denotes the function

$$\Gamma(x)=\sum_{n\geq 1}a_n^2\cos(nx). \tag{2}$$

note that $\Gamma\in C^8(\mathbf{R})$ (eight times differentiable) and is 2π periodic.

Under $\mathbf{Q}$, $\zeta(0,.)\equiv 0$ $\mathbf{Q}$-almost surely and ζ is Gaussian satisfying $E_{\mathbf{Q}}\{\zeta\}=0$, with covariance given by

$$E_{\mathbf{Q}}\{\zeta(t,x)\zeta(s,y)\}=(s\wedge t)\sum_{n\geq 1}a_n^2\cos(n(x-y))=(s\wedge t)\Gamma(x-y);$$

The moment fields are considered. Firstly, a priori bounds are calculated for

$$m_p^{(\epsilon)}(t,x_1,\ldots,x_p):=E_{\mathbf{Q}}\left\{u^{(\epsilon)}(t,x_1)\ldots u^{(\epsilon)}(t,x_p)\right\}. \tag{3}$$

for $t\in[0,T]$ where $T<+\infty$. These bounds are independent of ϵ. Secondly, the moment fields are shown to be Lipschitz, with the Lipschitz constant independent of ϵ. Thirdly, the main results of the article are the following theorems:

Theorem 1. *Let $u^{(\epsilon)}$ denote the solution to equation (1), where ζ satisfies hypothesis 1. Then for all non negative integer p,*

$$E_{\mathbf{Q}}\left\{u^{(\epsilon)2p+1}(t,x)\right\}=0 \qquad \forall t\geq 0,\ \epsilon>0,\ \forall x\in[0,2\pi]$$

and for all positive integer p and all $T<+\infty$,

$$\lim_{\epsilon\to 0}\sup_{0\leq t\leq T}\sup_{x\in[0,2\pi]}\left|E_{\mathbf{Q}}\left\{u^{(\epsilon)2p}(t,x)\right\}-(-\Gamma''(0))^p\prod_{j=1}^{p}(2j-1)t^p\right|=0. \tag{4}$$

and

Theorem 2. *There is an adapted function $u : \Omega \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ such that for each $p > 1$ and each $T < +\infty$*

$$\lim_{\epsilon \to 0} E_{\mathbf{Q}} \left\{ \int_0^T \int_0^{2\pi} \left| u^{(\epsilon)}(t,x) - u(t,x) \right|^p dx dt \right\} = 0.$$

For each $p > 0$, this function solves

$$\begin{cases} \partial_t u = -\frac{1}{2}(u^2)_x dt + \partial_t \zeta_x \\ u(0,x) \equiv 0 \end{cases}$$

These results ought to be of interest, following the results of E, Khanin, Mazel and Sinai in [1], showing existence of an invariant measure for this equation. The moments of the invariant measure constructed by E, Khanin, Mazel and Sinai are discussed in section 4, where it is shown that $E_{\mathbf{Q}} \left\{ \sup_{0 \leq x \leq 2\pi} |u(x)|^p \right\} < +\infty$ for each $p > 1$, where the distribution of u is the invariant measure for the equation.

Brief Outline From equation (1), it is proved (section 3) that the moment fields defined by equation (3) satisfy

$$\begin{aligned} &\frac{\partial}{\partial t} m_p^{(\epsilon)}(t; x_1, \ldots, x_p) = \frac{\epsilon}{2} \Delta_{\mathbf{x}} m_p^{(\epsilon)}(t; x_1, \ldots, x_p) \\ &\quad - \frac{1}{2} \sum_{j=1}^p \frac{\partial}{\partial x_j} m_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) + \sum_{1 \leq j < k \leq p} (-\Gamma''(x_j - x_k)) m_{p-2}(t; \hat{x}_j, \hat{x}_k) \end{aligned} \tag{5}$$

where $\frac{\partial}{\partial x_j}$ means derivative with respect to x_j (that is both appearances of x_j), $\hat{x}_j$ denotes that variable x_j has been omitted. This requires a Fubini theorem and the use of Itô's formula.

The non-linearity in the Burgers equation means the pth equation depends on the $p+1$ moment field. To show that there is a well defined solution to this system of equations that gives the moment fields, several a-priori bounds on moments of the solution to equation (1) and its derivatives have to be computed. The bounds on the moments and the first derivatives have to be uniform in ϵ to ensure that the limit, as ϵ tends to zero can be taken.

Theorem 11 gives a bound (independent of ϵ) on

$$E_{\mathbf{Q}} \left\{ \sup_{0 \leq \epsilon \leq 1} \sup_{0 \leq t \leq T} \sup_{0 \leq x \leq 2\pi} \left| u^{(\epsilon)}(t,x) \right|^p \right\}$$

and theorem 13 gives a bound (independent of ϵ) on

$$\sup_{0 < \epsilon \leq 1} \sup_{0 \leq t \leq T} \sup_{x_1, \ldots, x_p} \left| \frac{\partial}{\partial x_1} E_{\mathbf{Q}} \left\{ u^{(\epsilon)}(t, x_1) \ldots u^{(\epsilon)}(t, x_p) \right\} \right|.$$

After this, it is shown that the conditions are satisfied so that Itô's formula may be applied and orders of integration exchanged to show that the moment fields given by equation (3) satisfy equation (5).

The a priori upper bounds from theorems 11 and 13 are necessary for a-priori existence of solution to these moment equations; without them, there is no proof that the moment equations should have a solution. Next, an appropriate rescaling is carried out. Defining

$$\phi_p^{(\epsilon)}(t; x_1, \ldots, x_p) := \frac{m_p^{(\epsilon)}(t; \epsilon x_1, \ldots, \epsilon x_p) - m_p^{(\epsilon)}(t; 0, \ldots, 0)}{\epsilon}$$

and

$$\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) := m_p^{(\epsilon)}(t; \epsilon x_1, \ldots, \epsilon x_p)$$

gives

$$\frac{\partial}{\partial t}\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) = \frac{1}{2}\Delta_{\mathbf{x}}\phi_p^{(\epsilon)}(t; x_1, \ldots, x_p) \\ -\frac{1}{2}\sum_{j=1}^{p}\frac{\partial}{\partial x_j}\phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) + \sum_{1 \leq j < k \leq p}\mu_{p-2}^{(\epsilon)}(t; \hat{x}_j, \hat{x}_k)(-\Gamma''(\epsilon(x_j - x_k)). \tag{6}$$

The result in theorem 1 only requires the diagonal $m_p^{(\epsilon)}(t; 0, \ldots, 0) = E_{\mathbf{Q}}\left\{u^{(\epsilon)p}(t, x)\right\}$. This holds for all $x \in \mathbf{R}$ because the distribution of the random field and hence the distribution of $u^{(\epsilon)}(t, .)$ is spatially homogeneous. It is shown that for all $(x_1, \ldots, x_p) \in \mathbf{R}$ and all $T < +\infty$,

$$\lim_{\epsilon \to 0}\sup_{0 \leq t \leq T}|\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) - m_p^{(\epsilon)}(t; 0, \ldots, 0)| = 0,$$

and that there is a function $M_p(t)$ such that for all $T < +\infty$

$$\lim_{\epsilon \to 0}\sup_{0 \leq t \leq T}|m_p^{(\epsilon)}(t; 0, \ldots, 0) - M_p(t)| = 0$$

and such that for each $T < +\infty$, $M_p(t)$ is Lipschitz for $t \in [0, T]$. It is clear that

$$\lim_{\epsilon \to 0}|\Gamma''(\epsilon z) - \Gamma''(0)| = 0 \qquad \forall z \in \mathbf{R}.$$

For each $t \in \mathbf{R}$, $m_p^{(\epsilon)}(t; .)$ is Lipschitz in the space variable, uniformly in ϵ, from which it follows that $\phi_p^{(\epsilon)}(t; .)$ is Lipschitz in the space variable, uniformly in ϵ. The bounds on the growth, together with interpreting $\frac{1}{2}\Delta$ as the infinitesimal generator of the heat semigroup, enable the computations which show that the terms containing $\phi^{(\epsilon)}$ do not yield a contribution as $\epsilon \to 0$. From this, the results stated in theorem 1 are obtained. The uniform upper bound in ϵ from theorem 13 is necessary for the uniform bounds in ϵ on the Lipschitz constant for $\phi^{(\epsilon)}$ required to take the limit. By taking $\epsilon \to 0$ in equation (6), the proof of theorem 1 may be completed.

2 Bounds on the Moments

The following lemma is a necessary first step towards proving theorems 11 and 13.

Lemma 3. *Let β^j be independent standard Brownian motions with $\beta^j(0) = 0$, let*

$$S^j(t) = \sup_{0 \le s \le t} |\beta^j(t)|$$

and let a_j be numbers such that $\sum_j |a_j| < +\infty$. It follows that

$$E_{\mathbf{Q}}\left\{\exp\left\{\sum_j a_j S^j(t)\right\}\right\} \le \exp\left\{\frac{t}{2}\left(\sum_j a_j^2\right) + \sqrt{2t}(\sqrt{\log 2} + 2\sqrt{\pi})\left(\sum_j |a_j|\right)\right\}. \quad (7)$$

Proof of lemma 3 Firstly, from Revuz and Yor [4] page 55 proposition (1.8), if $S(t) = \sup_{0 \le s \le t} \beta(s)$ where β is a standard Brownian motion with $\beta(0) = 0$, such that for all $0 \le s \le u \le v \le t < +\infty$, $\beta(v) - \beta(u)$ is measurable with respect to $\mathcal{G}_t$ then, using $\mathbf{Q}\{.\}$ to denote the *probability* of an event with respect to $\mathbf{Q}$,

$$\mathbf{Q}\{S(1) \ge x\} \le \exp\left\{-\frac{x^2}{2}\right\}$$

and rescaling gives

$$\mathbf{Q}\{S(t) \ge x\} \le \exp\left\{-\frac{x^2}{2t}\right\}.$$

Let $\tilde{S}(t) = \sup_{0 \le s \le t} |\beta(s)| = \sup_{0 \le s \le t} \beta(s) \vee \sup_{0 \le s \le t}(-\beta(s))$. Note that

$$\begin{aligned} \mathbf{Q}\left\{\tilde{S}(t) \ge x\right\} &= \mathbf{Q}\left\{\left\{\sup_{0 \le s \le t} \beta(s) \ge x\right\} \cup \left\{\sup_{0 \le s \le t} (-\beta(s)) \ge x\right\}\right\} \\ &\le 1 \wedge 2\mathbf{Q}\left\{S(t) \ge x\right\} \le 1 \wedge 2\exp\left\{-\frac{x^2}{2t}\right\}. \end{aligned} \quad (8)$$

Note that

$$2e^{-x^2/2t} = 1 \quad \Rightarrow \quad x = \sqrt{2t \log 2}$$

(log means natural logarithm). It follows that for any $\alpha > 0$,

$$
\begin{aligned}
E_{\mathbf{Q}}\left\{e^{\alpha\tilde{S}(t)}\right\} &= \int_0^\infty \mathbf{Q}\left\{e^{\alpha\tilde{S}(t)} \geq y\right\} dy \\
&= \int_0^\infty \mathbf{Q}\left\{\tilde{S}(t) \geq \frac{1}{\alpha}\log y\right\} dy \\
&\leq e^{\alpha\sqrt{2t\log 2}} + \int_{e^{\alpha\sqrt{2t\log 2}}}^\infty \mathbf{Q}\left\{\tilde{S}(t) \geq \frac{1}{\alpha}\log y\right\} dy \\
&\leq e^{\alpha\sqrt{2t\log 2}} + 2\int_{e^{\alpha\sqrt{2t\log 2}}}^\infty e^{-\frac{1}{2\alpha^2 t}(\log y)^2} dy
\end{aligned}
$$

Substituting $x = \frac{\log y}{\alpha\sqrt{t}}$ so that $y = e^{\alpha\sqrt{t}x}$,

$$
\begin{aligned}
E_{\mathbf{Q}}\left\{e^{\alpha\tilde{S}(t)}\right\} &= e^{\alpha\sqrt{2t\log 2}} + 2\alpha\sqrt{t}\int_{\sqrt{2\log 2}}^\infty e^{-\frac{x^2}{2}+\alpha\sqrt{t}x} dx \\
&= e^{\alpha\sqrt{2t\log 2}} + 2\alpha\sqrt{t}e^{\alpha^2 t/2}\int_{\sqrt{2\log 2}-\alpha\sqrt{t}}^\infty e^{-\frac{x^2}{2}} dx \\
&\leq \exp\left\{\alpha\sqrt{2t\log 2}\right\} + 2\alpha\sqrt{2\pi t}\exp\left\{\frac{\alpha^2 t}{2}\right\}.
\end{aligned}
$$

Now note that for any non negative numbers a, b, c, α,

$$
e^{a\alpha} + b\alpha e^{c\alpha^2} \leq e^{a\alpha + c\alpha^2}(1 + b\alpha) \leq e^{(a+b)\alpha + c\alpha^2}.
$$

It follows that, for any $\alpha \geq 0$,

$$
E_{\mathbf{Q}}\left\{e^{\alpha\tilde{S}(t)}\right\} \leq \exp\left\{\sqrt{2t}(\sqrt{\log 2} + 2\sqrt{\pi})\alpha + \frac{\alpha^2 t}{2}\right\}.
$$

It follows that

$$
\begin{aligned}
E_{\mathbf{Q}}\left\{e^{\sum_j a_j S^j(t)}\right\} &= \prod_{j=1}^\infty E_{\mathbf{Q}}\left\{e^{a_j S^j(t)}\right\} \\
&\leq \exp\left\{\frac{t}{2}\left(\sum_j a_j^2\right) + (\sqrt{2t}(\sqrt{\log 2} + 2\sqrt{\pi}))\left(\sum_j |a_j|\right)\right\}
\end{aligned}
$$

and lemma 3 is proved. $\square$

The following bound will also be useful in the sequel.

Lemma 4. *Let β^j be independent standard Brownian motions with $\beta^j(0) = 0$ and let*

$$S^j(t) = \sup_{0 \le s \le t} \left|\beta^j(s)\right|.$$

Let a_j be real numbers such that $\sum_j |a_j| < +\infty$. Let

$$G(p) = \int_0^\infty y^p \exp\left\{-\frac{y^2}{2}\right\} dy = 2^{(p-1)/2}\Gamma_{Eu}\left(\frac{p+1}{2}\right), \tag{9}$$

where Γ_{Eu} denotes the Euler Gamma function $\Gamma_{Eu}(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$. It holds that

$$E_{\mathbf{Q}}\left\{\left|\sum_j a_j S^j(t)\right|^p\right\} \le \left(\sum_j |a_j|\right)^p t^{p/2}\left((2\log 2)^{p/2} + 2pG(p-1)\right). \tag{10}$$

Proof Using inequality (8), it follows that

$$\mathbf{Q}\{S^j(t) \ge x\} \le 1 \wedge 2\exp\left\{-\frac{x^2}{2t}\right\}.$$

Provided $0 < \sum_j |a_j| < +\infty$, a standard application of Jensen's inequality gives, for $p \ge 1$,

$$\begin{aligned} E_{\mathbf{Q}}\left\{\left|\sum_j a_j S^j(t)\right|^p\right\} &\le E_{\mathbf{Q}}\left\{\left(\sum_j |a_j| S^j(t)\right)^p\right\} \\ &= \left(\sum_j |a_j|\right)^p E_{\mathbf{Q}}\left\{\left(\sum_j \frac{|a_j|}{\sum_j |a_j|} S_j(t)\right)^p\right\} \\ &\le \left(\sum_j |a_j|\right)^{p-1} \sum_j |a_j| E_{\mathbf{Q}}\left\{S^j(t)^p\right\} \end{aligned}$$

and

$$\begin{aligned} E_{\mathbf{Q}}\left\{S^j(t)^p\right\} &= \int_0^\infty \mathbf{Q}\left\{S^j(t)^p \ge x\right\} dx = \int_0^\infty \mathbf{Q}\left\{S^j(t) \ge x^{1/p}\right\} dx \\ &\le \int_0^\infty \left(1 \wedge 2\exp\left\{-\frac{x^{2/p}}{2t}\right\}\right) dx \\ &= (2t\log 2)^{p/2} + 2\int_{(2t\log 2)^{p/2}}^\infty e^{-x^{2/p}/2t} dx \\ &= (2t\log 2)^{p/2} + 2pt^{p/2}\int_{(2\log 2)^{1/2}}^\infty z^{p-1}e^{-z^2/2} dz \\ &\le t^{p/2}\left((2\log 2)^{p/2} + 2pG(p-1)\right) \end{aligned}$$

which gives

$$E_{\mathbf{Q}}\left\{\left|\sum_j a_j S^j(t)\right|^p\right\} \leq \left(\sum_j |a_j|\right)^p t^{p/2}\left((2\log 2)^{p/2} + 2pG(p-1)\right)$$

as advertised. $\square$

Lemma 5. *For $p > 2$, $T > 0$, let $\mathcal{S}_{p,T}$ denote the space of functions $f : \mathbf{R}_+ \times \mathbf{R} \times \Omega \to \mathbf{R}$, 2π periodic in the space (second) variable, such that for each $s \in \mathbf{R}$, $f : [0,s] \times \mathbf{R} \times \Omega \to \mathbf{R}$ is measurable with respect to $\mathcal{B}([0,s]) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{G}_s$ (where $\mathcal{B}$ denotes the Borel σ-algebra) and such that*

$$|||f|||_{p,T} := \left(\sup_{0\leq t\leq T} \frac{1}{2\pi}\int_0^{2\pi} E_{\mathbf{Q}}\left\{|f(t,x)|^p\right\} dx\right)^{1/p} < +\infty \tag{11}$$

Let

$$\mathcal{S}_p = \cap_{T>0}\mathcal{S}_{p,T}$$

and let

$$\mathcal{S}_p^* = \{f \in \mathcal{S}_p\, | f(t,.) \in C^2(\mathbf{R})\ \forall t < +\infty\ \mathbf{Q} - a.s.\} \tag{12}$$

For each $\epsilon > 0$, $p \in (0,+\infty)$, there exists a unique solution such that $U^{(\epsilon)} \in \mathcal{S}_p$ to the equation

$$\begin{cases} \partial_t U^{(\epsilon)} = \frac{\epsilon}{2} U^{(\epsilon)}_{xx} dt - \frac{1}{\epsilon} U^{(\epsilon)} \circ \partial_t \zeta \\ U^{(\epsilon)}(0,x) \equiv 1, \end{cases} \tag{13}$$

where the $\circ$ denotes stochastic integration in the Stratonovich sense. The solution $U^{(\epsilon)}$ satisfies the following regularity: for all $T < +\infty$, there is a version such that almost surely, $U^{(\epsilon)} \in C^{0,\alpha}([0,T] \times \mathbf{R})$ (Hölder continuous of order α) for all $\alpha < \frac{1}{2}$ and for each $t \in [0,T]$, $U^{(\epsilon)}(t,.) \in C^{3,\gamma}(\mathbf{R})$ (three times differentiable, third derivative Hölder continuous of order γ) for all $\gamma < 1$, where for each $x \in \mathbf{R}$, $U^{(\epsilon)}(.,x), U^{(\epsilon)}_x(.,x), U^{(\epsilon)}_{xx}(.,x), U^{(\epsilon)}_{xxx}(.,x)$ are Hölder continuous in the time variable of all orders less than $1/2$.

Proof Existence and uniqueness of solution to equation (13) is standard and may be found (for example) in Kunita [3]. Kunita also shows that if $\Gamma \in C^{2n}$ then, almost surely, $U^{(\epsilon)}(t,.) \in C^{n-1,\gamma}$ (that is $n-1$ times differentiable, $n-1$th derivative Hölder continuous of order γ) for all $\gamma < 1$ in the space variable and if $n \geq 2$, then $U^{(\epsilon)}(.,x)$ and its first n derivatives are Hölder continous of all orders less than $\frac{1}{2}$ in the time variable. Kunita's results are more extensive; those stated above are the only ones needed here.

To keep this article relatively self contained, an outline of the proof is sketched here. The Itô formulation of the mild form of equation (13) is

$$
\begin{aligned}
&U^{(\epsilon)}(t,x) = \\
&\quad 1+\frac{1}{\epsilon}\sum_{n\geq 1} a_n\left(\int_0^t P_{t-s}(U^{(\epsilon)}(s,.)\cos(n.))(x)d\beta_s^{1n} + \int_0^t P_{t-s}(U^{(\epsilon)}(s,.)\sin(n.))(x)d\beta_s^{2n}\right) \\
&\quad +\frac{\Gamma(0)}{2\epsilon^2}\int_0^t P_{t-s}U^{(\epsilon)}(s,x)ds
\end{aligned} \tag{14}
$$

where for a bounded measurable function f, P_t is defined such that

$$
P_t f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\epsilon t}} e^{-y^2/2\epsilon t} f(x+y)dy. \tag{15}
$$

From Revuz and Yor page 137 definition (2.1) and theorem (2.2), it is sufficient (but not necessary) that $E_{\mathbf{Q}}\left\{\int_0^t f_s^2 ds\right\} < +\infty$ for an adapted measurable function f to ensure that the stochastic integral $\int_0^t f_s d\beta_s^{jn}$ is well defined. Set $U^{(\epsilon,0)} = U^{(\epsilon)}$, $f_{1k}(x) = \frac{d^k}{dx^k}\cos(x)$ and $f_{2k}(x) = \frac{d^k}{dx^k}\sin(x)$. Recall the standard result that for two functions f and g, both n times differentiable, the nth derivative of the product satisfies

$$
(fg)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)} g^{(n-j)}.
$$

For $a \geq 1$, let $U^{(\epsilon,a)}$ satisfy

$$
\begin{aligned}
U^{(\epsilon,a)}(t,x) &= \frac{1}{\epsilon}\sum_{k=0}^{a}\binom{a}{k}\sum_{n\geq 1} n^k a_n\left(\int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{1k}(n.)\right)(x)d\beta_s^{1n}\right. \\
&\quad \left.+\int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{2k}(n.)\right)(x)d\beta_s^{2n}\right) + \frac{\Gamma(0)}{2\epsilon^2}\int_0^t P_{t-s}U^{(\epsilon,a)}(s,x)ds
\end{aligned} \tag{16}
$$

Suppose that existence and uniqueness of solution in $\mathcal{S}_p$ for all $p \geq 2$ have been established for $b = 0,1,\ldots,a-1$. Consider the iterative sequence: $U_0^{(\epsilon,0)} \equiv 1$ and $U_0^{(\epsilon,a)} \equiv 0$ for $a = 1,2,3$ and

$$
\begin{aligned}
&U_{m+1}^{(\epsilon,a)}(t,x) = P_t U^{(\epsilon,a)}(0,x) \\
&\quad +\frac{1}{\epsilon}\sum_{k=1}^{a}\binom{a}{k}\sum_{n\geq 1} n^k a_n\left(\int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{1k}(n.)\right)(x)d\beta_s^{1n}\right. \\
&\quad \left.+\int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{2k}(n.)\right)(x)d\beta^{2n}\right) + \frac{\Gamma(0)}{2\epsilon^2}\int_0^t P_{t-s}U_m^{(\epsilon,a)}(s,x)ds \\
&\quad +\frac{1}{\epsilon}\sum_{n\geq 1} a_n\left(\int_0^t P_{t-s}\left(U_m^{(\epsilon,a)}(s,.)\cos(n.)\right)(x)d\beta_s^{1n} + \int_0^t P_{t-s}\left(U_m^{(\epsilon,a)}(s,.)\sin(n.)\right)(x)d\beta^{2n}\right)
\end{aligned} \tag{17}
$$

Set $D_m^{(\epsilon)} = U_m^{(\epsilon)} - U_{m-1}^{(\epsilon)}$ for $m \geq 1$ and $D_m^{(\epsilon,a)} = U_m^{(\epsilon,a)} - U_{m-1}^{(\epsilon,a)}$. Then, using $D_m^{(\epsilon)} = D_m^{(\epsilon,0)}$, it follows that for $a = 0,1,2,3$ and $m \geq 1$,

$$D_{m+1}^{(\epsilon,a)}(t,x) = \frac{\Gamma(0)}{2\epsilon^2}\int_0^t P_{t-s}D_m^{(\epsilon,a)}ds \tag{18}$$
$$+\frac{1}{\epsilon}\sum_{n\geq 1} a_n\left(\int_0^t P_{t-s}\left(D_m^{(\epsilon,a)}(s,.)\cos(n.)\right)(x)d\beta_s^{1n} + \int_0^t P_{t-s}\left(D_m^{(\epsilon,a)}(s,.)\sin(n.)\right)(x)d\beta^{2n}\right).$$

while

$$D_1^{(\epsilon)}(t,x) = \frac{\Gamma(0)}{2\epsilon^2}t + \frac{1}{\epsilon}\sum_{n\geq 1} a_n\left(\int_0^t P_{t-s}\cos(nx)d\beta_s^{1n} + P_{t-s}\sin(nx)d\beta_s^{2n}\right)$$

and, for $a = 1, 2, 3$,

$$D_1^{(\epsilon,a)}(t,x) = \frac{1}{\epsilon}\sum_{k=1}^{a}\left(\begin{array}{c} a \\ k \end{array}\right)\sum_{n\geq 1} n^k a_n\left(\int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{1k}(n.)\right)(x)d\beta_s^{1n}\right.$$
$$\left. + \int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{2k}(n.)\right)(x)d\beta^{2n}\right),$$

so that for each $p \geq 2$ there is a constant $C_p < +\infty$ such that (using $\Gamma(0) = \sum_{n\geq 1} a_n^2$)

$$E_{\mathbf{Q}}\left\{|D_1^{(\epsilon)}(t,x)|^p\right\} \leq C_p\left(\frac{\Gamma(0)}{2\epsilon^2}t\right)^p + \frac{C_p t^{p/2}}{\epsilon^p}\Gamma(0)^{p/2}$$

and

$$E_{\mathbf{Q}}\left\{|D_1^{(\epsilon,a)}(t,x)|^p\right\} \leq \frac{C_p}{\epsilon^p}\left(\sum_{n\geq 1} n^6 a_n^2\right)^{p/2} t^{(p/2)-1}\sum_{k=1}^{a}\int_0^t E_{\mathbf{Q}}\left\{|U^{(\epsilon,a-k)}(s,x)|^p\right\}ds \quad a = 1, 2, 3$$

From equation (18), it follows that

$$D_{m+1}^{(\epsilon,a)2p}(t,x) \leq 2^{2p}\left(\left(\frac{1}{\epsilon}\sum_{n\geq 1} n^a a_n\left(\int_0^t P_{t-s}(D_m^{(\epsilon,a)}(s,.)\cos(n.))(x)d\beta_s^{1n}\right.\right.\right.$$
$$\left.\left.\left. + \int_0^t P_{t-s}(D_m^{(\epsilon,a)}(s,.)\sin(n.))(x)d\beta_s^{2n}\right)\right)^{2p} + \left(\frac{\Gamma(0)}{2\epsilon^2}\int_0^t P_{t-s}D_m^{(\epsilon,a)}(s,.)(x)ds\right)^{2p}\right).$$

In the following, the constant K may change from line to line. It denotes a positive finite value. Let

$$M_r = \frac{1}{\epsilon}\sum_{n\geq 1} n^a a_n\left(\int_0^r P_{t-s}(D_m^{(\epsilon,a)}(s,.)\cos(n.))(x)d\beta_s^{1n} + \int_0^r P_{t-s}(D_m^{(\epsilon,a)}(s,.)\sin(n.))(x)d\beta_s^{2n}\right)$$

defined on the time interval $r \in [0,t]$. From equation (17), it is clear (under the assumption that $U^{(\epsilon,b)} \in \mathcal{S}_p$ for each $b < a$) that $U_m^{(\epsilon,a)} \in \mathcal{S}_p$ for each $m < +\infty$, from which it follows directly that $D_m^{(\epsilon,a)} \in \mathcal{S}_p$ for each $m < +\infty$ and hence that M is a local martingale. Note that

$$\langle M \rangle_r = \frac{1}{\epsilon^2} \sum_{n \geq 1} n^{2a} a_n^2 \int_0^r \left((P_{t-s}(D_m^{(\epsilon,a)}(s,.) \cos(n.))(x))^2 + (P_{t-s}(D_m^{(\epsilon,a)}(s,.) \sin(n.))(x))^2 \right) ds.$$

It follows from the Burkholder-Davis-Gundy inequality that

$$E_{\mathbf{Q}} \left\{ D_{m+1}^{(\epsilon,a)2p}(t,x) \right\} \leq K E_{\mathbf{Q}} \left\{ \left(\frac{1}{\epsilon^2} \sum_{n \geq 1} n^{2a} a_n^2 \left(\int_0^t (P_{t-s} D_m^{(\epsilon,a)}(s,.) \cos(n.)(x))^2 ds \right.\right.\right.$$
$$\left.\left.\left. + \int_0^t (P_{t-s} D_m^{(\epsilon,a)}(s,.) \sin(n.)(x)))^2 ds \right)^p \right\} + K E_{\mathbf{Q}} \left\{ \left(\frac{\Gamma(0)}{2\epsilon^2} \int_0^t P_{t-s} D_m^{(\epsilon,a)}(s,x) ds \right)^{2p} \right\} \right).$$

Let $\|.\|_p$ denote the norm defined by

$$\|f\|_p(t) = \sup_{0 \leq s \leq t} \left(\frac{1}{2\pi} \int_0^{2\pi} E_{\mathbf{Q}} \left\{ |f(s,x)| \right\} dx \right)^{1/p} \tag{19}$$

Then straightforward applications of Hölder's inequality give, for $p \geq 2$,

$$\|D_{m+1}^{(\epsilon,a)}\|_p^p(t) \leq K \frac{\left(\sum_{n \geq 1} n^{2a} a_n^2 \right)^{p/2}}{\epsilon^p} t^{(p/2)-1} \int_0^t \|D_m^{(\epsilon,a)}\|_p^p(s) ds + K \frac{\Gamma^p(0)}{\epsilon^{2p}} t^{p-1} \int_0^t \|D_m^{(\epsilon,a)}\|_p^p(s) ds.$$

Note that, since the field is spatially homogeneous, $\|D_m^{(\epsilon,a)}\|_p^p(t) = E_{\mathbf{Q}} \left\{ |D_m^{(\epsilon,a)}(t,x)|^p \right\}$. Therefore, for all $t \in [0,T]$, for $T < +\infty$,

$$\|D_{m+1}^{(\epsilon,a)}\|_p^p(t) \leq C(T) \int_0^t \|D_m^{(\epsilon,a)}\|_p^p(s) ds$$

where

$$C(T) = K \frac{\left(\sum_{n \geq 1} n^{2a} a_n^2 \right)^{p/2}}{\epsilon^p} T^{(p/2)-1} + K \frac{\Gamma(0)^p}{\epsilon^{2p}} T^{p-1} \int_0^t E_{\mathbf{Q}} \left\{ |D_1^{(\epsilon)}(s,x)|^p \right\} ds,$$

so that, for $0 \leq t \leq T$,

$$\|D_m^{(\epsilon,a)}\|_p(t) \leq \left(\frac{(C(T)t)^{m-1}}{(m-1)!} \right)^{1/p} \left(\int_0^t \|D_1^{(\epsilon,a)}\|_p^p(s) ds \right)^{1/p}.$$

Since $\sum_{n \geq 1} n^8 a_n^2 < +\infty$, it follows that $\sum_{n \geq 1} n^{2a} a_n^2 < +\infty$ for $a = 0,1,2,3$, it follows inductively, that $\|D_m^{(\epsilon,a)}\|_p(t)$ is summable. This is clearly true for $a = 0$, which implies that

$$\sup_{0\le s\le T} E_{\mathbf{Q}}\left\{|D_1^{(\epsilon)}(s,x)|^p\right\} < +\infty$$

and hence is true inductively for $a = 1, 2, 3$. Existence and uniqueness for solutions to equation (14) in $\mathcal{S}_p$ for all $p \in [2, +\infty)$ now follows directly by standard Gronwall arguments.

Now set

$$C_{a,p}(t) = \sup_{0\le s\le t} E_{\mathbf{Q}}\left\{|U^{(\epsilon,a)}(t,x)|^p\right\}$$

The preceeding gives that $\sup_{0\le t\le T} C_{a,p}(t) < +\infty$ for $a = 0, 1, 2, 3$.
Now set

$$V^{(a,h)}(t,x) = \frac{U^{(\epsilon,a)}(t,x+h) - U^{(\epsilon,a)}(t,x-h)}{2h}, \tag{20}$$

$f_{1k}(x) = \frac{d^k}{dx^k}\cos(x)$ and $f_{2k}(x) = \frac{d^k}{dx^k}\sin(x)$. Then, since $U^{(\epsilon,0)}(0,.) \equiv 1$ and $U^{(\epsilon,a)}(0,.) \equiv 0$ for $a = 1, 2, 3$, it follows that $V^{(a,h)}$ satisfies

$$\begin{aligned}
V^{(a,h)}(t,x) = {} & \frac{1}{\epsilon}\sum_{k=0}^{a}\begin{pmatrix} a \\ k \end{pmatrix}\sum_{n\ge 1} n^k a_n \left(\int_0^t P_{t-s}(V^{(a-k,h)}(s,.)f_{1k}(n(.+h)))(x)d\beta_s^{1n}\right. \\
& \left. + \int_0^t P_{t-s}(V^{(a-k,h)}(s,.)f_{2k}(n(.+h)))(x)d\beta_s^{2n}\right) + \frac{\Gamma(0)}{2\epsilon^2}\int_0^t P_{t-s}V^{(a,h)}(s,.)ds \\
& + \frac{1}{\epsilon}\sum_{k=0}^{a}\begin{pmatrix} a \\ k \end{pmatrix} \\
& \times \sum_{n\ge 1} n^k a_n \left(\int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.-h)\left(\frac{f_{1k}(n(.+h) - f_{1k}(n(.-h))}{2h}\right)\right)(x)d\beta_s^{1n}\right. \\
& \left. + \int_0^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.-h)\left(\frac{f_{2k}(n(.+h)) - f_{2k}(n(.-h))}{2h}\right)\right)(x)d\beta_s^{2n}.\right.
\end{aligned} \tag{21}$$

Set

$$K_{a,p,h}(t) = \sup_{0\le s\le t} E_{\mathbf{Q}}\left\{|V^{(a,h)}(t,x)|^p\right\}. \tag{22}$$

For all $2 \le p < +\infty$, elementary arguments give the existence of a constant $c_1(p,a,T) < +\infty$ such that

$$
\begin{aligned}
K_{a,p,h}(t) \quad \leq \quad & \frac{c_1(p,a,T)}{\epsilon^p}\left(\sum_{n\geq 1} n^6 a_n^2\right)^{p/2} t^{(p/2)-1}\sum_{k=0}^{a}\int_0^t K_{a-k,p,h}(s)ds \\
& + \frac{c_1(p,a,T)}{\epsilon^p}\left(\sum_{n\geq 1} n^8 a_n^2\right)^{p/2} t^{(p/2)-1}\sum_{k=0}^{a}\int_0^t C_{a-k,p}(s)ds \\
& + \frac{c_1(p,a,T)}{\epsilon^{2p}}\Gamma(0)^p t^{p-1}\int_0^t K_{a,p,h}(s)ds
\end{aligned}
$$

from which it follows that for all $T < +\infty$, $2 \leq p < +\infty$ and $a = 0,1,2,3,4$, $C_{a,p}(T) < +\infty$ and for $a = 0,1,2,3$, $\tilde{K}_{a,p}(T) := \sup_{h>0} K_{a,p,h}(T) < +\infty$. These bounds enable Kolmogorov's criterion to be applied to the space variable.

The following computations enable the appropriate Hölder continuity to be proved for the time variable. Set

$$
\begin{aligned}
I^{(\epsilon,a)}(r,t;x) \quad = \quad & \frac{1}{\epsilon}\sum_{k=0}^{a}\binom{a}{k}\sum_{n\geq 1} n^k a_n \left(\int_r^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{1k}(n.)\right)(x)d\beta_s^{1n}\right. \\
& \left. + \int_r^t P_{t-s}\left(U^{(\epsilon,a-k)}(s,.)f_{2k}(n.)\right)(x)d\beta_s^{2n}\right)
\end{aligned}
$$

and note that, for $a = 0,1,2,3$

$$
\begin{aligned}
& U^{(\epsilon,a)}(t+h,x) - U^{(\epsilon,a)}(t,x) \\
& \quad = (P_h - I)U^{(\epsilon,a)}(t;x) + I^{(\epsilon,a)}(t,t+h;x) + \frac{\Gamma(0)}{2\epsilon^2}\left(\int_t^{t+h} P_{t+h-s}U^{(\epsilon,a)}(s,x)ds\right).
\end{aligned}
$$

Let $p_t(z) = \frac{1}{\sqrt{2\pi t}}e^{-z^2/2t}$ and note that $\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{|z|}{\sqrt{t}}p_t(z)dz = 1$. From equation (15), it follows that

$$
\begin{aligned}
& E_{\mathbf{Q}}\left\{\left|(P_h - P_0)U^{(\epsilon,a)}(t,x)\right|^p\right\} = E_{\mathbf{Q}}\left\{\left|\int_{-\infty}^{\infty} p_{\epsilon h}(x-y)\left(U^{(\epsilon,a)}(t,y) - U^{(\epsilon,a)}(t,x)\right)dy\right|^p\right\} \\
& \quad = \left(\frac{2\epsilon h}{\pi}\right)^{p/2} E_{\mathbf{Q}}\left\{\left|\int_{-\infty}^{\infty}\frac{|x-y|}{\sqrt{\epsilon h}}\sqrt{\frac{\pi}{2}}p_{\epsilon h}(x-y)\left(\frac{U^{(\epsilon,a)}(t,y) - U^{(\epsilon,a)}(t,x)}{|x-y|}\right)dy\right|^p\right\} \\
& \quad \leq \left(\frac{2\epsilon h}{\pi}\right)^{p/2}\int_{-\infty}^{\infty}\frac{|x-y|}{\sqrt{\epsilon h}}\sqrt{\frac{\pi}{2}}p_{\epsilon h}(x-y)E_{\mathbf{Q}}\left\{\left|\frac{U^{(\epsilon,a)}(t,y) - U^{(\epsilon,a)}(t,x)}{x-y}\right|^p\right\}dy \\
& \quad \leq \left(\frac{2\epsilon h}{\pi}\right)^{p/2}\tilde{K}_{a,p}(T).
\end{aligned}
$$

Straightforward bounds give, for $p \geq 2$, a constant $C_p < +\infty$ such that

$$
\begin{aligned}
E_{\mathbf{Q}}\left\{|I^{(\epsilon,a)}(t,t+h;x)\right\} &\leq \frac{C_p}{\epsilon^p}h^{(p/2)-1}\left(\sum_{n\geq 1}n^6a_n^2\right)^{p/2}\sum_{k=0}^{a}\int_t^{t+h}E_{\mathbf{Q}}\left\{\left|U^{(\epsilon,a)}(s,x)\right|^p\right\}ds \\
&\leq h^{p/2}\frac{C_p}{\epsilon^p}\left(\sum_{n\geq 1}n^6a_n^2\right)^{p/2}\sum_{k=0}^{a}C_{p,a}(T).
\end{aligned}
$$

Finally, for $p \geq 2$ and $0 \leq t \leq t+h \leq T$,

$$
E_{\mathbf{Q}}\left\{\left|\frac{\Gamma(0)}{2\epsilon^2}\left(\int_t^{t+h}P_{t+h-s}U^{(\epsilon,a)}(s,x)ds\right)\right|^p\right\} \leq \left(\frac{\Gamma(0)}{2\epsilon^2}\right)^p h^p C_{p,a}(T).
$$

It follows that for all $p \in (2,+\infty)$ and $T < +\infty$, $a = 0,1,2,3$, there exists a constant $c_2(p,\epsilon,a,T) < +\infty$ such that for all $x \in \mathbf{R}$ and all $t \in [0,T-h]$,

$$
E_{\mathbf{Q}}\left\{\left|U^{(\epsilon,a)}(t+h,x) - U^{(\epsilon,a)}(t,x)\right|^{2p}\right\} \leq c_2(p,\epsilon,a,T)h^p. \tag{23}
$$

By the inequality (23) together with equation (20) and inequality (22), it follows directly by Kolmogorov's criterion that there is a version such that almost surely, for $a = 0,1,2,3$, $U^{(\epsilon,a)}$ is a Hölder continuous function of all orders less than $\frac{1}{2}$ on $[0,T]\times\mathbf{R}$. Furthermore, for all $t \in [0,T]$ $U^{(\epsilon,a)}(t,.) \in C^{0,\gamma}(\mathbf{R})$ for all $\gamma < 1$.

Furthermore, for $a = 0,1,2,3$, setting $\tilde{U}^{(\epsilon,a)}(t,x) = U^{(\epsilon,a)}(t,0) + \int_0^x U^{(\epsilon,a+1)}(t,y)dy$, it is straightforward to show that $\tilde{U}^{(\epsilon,a)}$ satisfies equation (14) and hence, since the solution to the equation is unique, that $\tilde{U}^{(\epsilon,a)} = U^{(\epsilon,a)}$. It therefore follows that $U^{(\epsilon,1)} = U_x^{(\epsilon)}$, $U^{(\epsilon,2)} = U_{xx}^{(\epsilon)}$ and $U^{(\epsilon,3)} = U_{xxx}^{(\epsilon)}$. It follows that for all $T < +\infty$, $U^{(\epsilon)}$ has a version which is three times differentiable in the space variable and such that $U_{xxx}^{(\epsilon)} \in C^{0,\alpha}([0,T]\times\mathbf{R})$ for all $\alpha < \frac{1}{2}$ and $U_{xxx}^{(\epsilon)}(t,.) \in C^{0,\gamma}(\mathbf{R})$ for all $\gamma < 1$ and all $t \in [0,T]$.

For $0 < p < 2$, existence is straightforward since a straightforward application of Hölder's inequality gives that $\mathcal{S}_{p_1} \in \mathcal{S}_{p_2}$ for $p_2 < p_1$. For uniqueness, consider $0 < p < 2$ and suppose that $U^{(\epsilon)}$ and $V^{(\epsilon)}$ are two solutions to equation (13) $\mathcal{S}_p$. Then for each $T < +\infty$,

$$
|||U^{(\epsilon)} - V^{(\epsilon)}|||_{p,T} \leq |||U^{(\epsilon)} - V^{(\epsilon)}|||_{2,T} = 0
$$

establishing uniqueness in $\mathcal{S}_p$. The proof of lemma 5 is complete. $\square$

A Kacs - Feynmann representation may be constructed for the solution of equation (13). To do so, a standard Wiener process, independent of ζ is introduced. The sample paths of this Wiener process are denoted by w, and $w_0 = 0$. The notation $w^{(\epsilon)} := \sqrt{\epsilon}w$ is used; this is a Wiener process, independent of ζ with diffusion coefficient ϵ. The probability measure associated to this Wiener

process is denoted $\mathbf{P}$ and the expectation operator with respect to this Wiener process $E_{\mathbf{P}}[.]$. The Kacs Feynman representation is

$$\begin{aligned} & U^{(\epsilon)}(t,x) \\ &= E_{\mathbf{P}}\left[\exp\left\{-\frac{1}{\epsilon}\left(\sum_{n\geq 1} a_n\left(\int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{1n} + \int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{2n}\right)\right)\right\}\right], \end{aligned} \tag{24}$$

where $w^{(\epsilon)}$ denotes a Browian motion with respect to $\mathbf{P}$, $w_0^{(\epsilon)} = 0$, with diffusion coefficient ϵ.

Lemma 6 (Bounds)**.** *Let $S^{j,n}(t) = \sup_{0\leq s\leq t}|\beta_s^{j,n}|$. Then*

$$\inf_{x\in[0,2\pi)}\inf_{t\in[0,T]} U^{(\epsilon)}(t,x) \geq \exp\left\{-\frac{1}{\epsilon}\sum_{n\geq 1}|a_n|(1+\epsilon n^2 T)(S_T^{1,n}(T)+S_T^{2,n}(T))\right\}. \tag{25}$$

It follows that, for any $T < +\infty$,

$$\begin{aligned} & \epsilon E_{\mathbf{Q}}\left\{\inf_{t\in[0,T]}\inf_{x\in[0,2\pi)}\log U^{(\epsilon)}(t,x)\right\} \\ & \qquad \geq -2(\sqrt{2\log 2}+\sqrt{2\pi})\sqrt{T}\left(\sum_{n\geq 1}|a_n| + \epsilon T\sum_{n\geq 1} n^2|a_n|^2\right) > -\infty \end{aligned} \tag{26}$$

and for each $\epsilon > 0$, each $0 < p < +\infty$ and each $T < +\infty$ there exists a constant $K(p,\epsilon,T)$ such that

$$\sup_{0\leq t\leq T}\sup_{0\leq x\leq 2\pi} E_{\mathbf{Q}}\left\{\left|\frac{\epsilon U_x^{(\epsilon)}(t,x)}{U^{(\epsilon)}(t,x)}\right|^p\right\} \leq K(p,\epsilon,T) < +\infty. \tag{27}$$

Proof Applying Jensen's inequality to equation (24), where P_t is defined as in equation (15), note that

$$\frac{\partial}{\partial s}P_{t-s}\cos(nx) = -\frac{\epsilon}{2}\frac{\partial^2}{\partial x^2}P_{t-s}\cos(nx) = \frac{n^2\epsilon}{2}P_{t-s}\cos(nx), \qquad \frac{\partial}{\partial s}P_{t-s}\sin(nx) = \frac{n^2\epsilon}{2}P_{t-s}\sin(nx)$$

so that

$$\begin{aligned} U^{(\epsilon)}(t,x) &\geq e^{-\frac{1}{\epsilon}\sum_{n\geq 1} a_n(\int_0^t P_{t-s}\cos(nx)d\beta_s^{1n} + \int_0^t P_{t-s}\sin(nx)d\beta_s^{2n})} \\ &= e^{-\frac{1}{\epsilon}\sum_{n\geq 1} a_n(\beta_t^{1n}\cos(nx)+\beta_t^{2n}\sin(nx)) - \frac{1}{2}\sum_{n\geq 1} n^2 a_n(\int_0^t \beta_s^{1n}P_{t-s}\cos(nx)ds + \int_0^t \beta_s^{2n}P_{t-s}\sin(nx)ds)} \\ &\geq e^{-\frac{1}{\epsilon}\sum_{n\geq 1}|a_n|(1+\epsilon n^2 T)(S^{1n}(T)+S^{2n}(T))} \end{aligned}$$

and inequality (25) follows. The inequality (6) follows from inequality (8), which gives

$$
\begin{aligned}
E_{\mathbf{Q}}\{S^{jn}(T)\} &= \int_0^\infty \mathbf{Q}\{S^{jn}(T) \geq x\} dx \\
&\leq \int_0^\infty 1 \wedge 2e^{-x^2/2T} dx \\
&= \sqrt{2T \log 2} + 2 \int_{\sqrt{2T \log 2}}^\infty e^{-x^2/2T} dx \\
&\leq \sqrt{2T \log 2} + \sqrt{2\pi T}.
\end{aligned}
$$

Lastly, inequality (27 is considered. It follows from taking a derivative with respect to x in equation (24) that

$$
\begin{aligned}
&\epsilon U_x^{(\epsilon)}(t,x) \\
&\quad = E_{\mathbf{P}}\left[\exp\left\{-\frac{1}{\epsilon}\left(\sum_{n\geq 1} a_n \left(\int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{1n} + \int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{2n}\right)\right)\right\}\right. \\
&\qquad \left.\times \sum_{n\geq 1} n a_n \left(\int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{1n} - \int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{2n}\right)\right]
\end{aligned}
$$

and an application of Hölder's inequality gives

$$
\begin{aligned}
&\left|\epsilon U_x^{(\epsilon)}(t,x)\right| \\
&\quad \leq E_{\mathbf{P}}\left[\exp\left\{-\frac{2}{\epsilon}\left(\sum_{n\geq 1} a_n \left(\int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{1n} + \int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{2n}\right)\right)\right\}\right]^{1/2} \\
&\qquad \times E_{\mathbf{P}}\left[\left(\sum_{n\geq 1} n a_n \left(\int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{1n} - \int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right) d\beta_s^{2n}\right)\right)^2\right]^{1/2}
\end{aligned}
$$

From this, for $p > 0$,

$$
\begin{aligned}
&E_{\mathbf{Q}}\left\{\left|\frac{\epsilon U_x^{(\epsilon)}(t,x)}{U^{(\epsilon)}(t,x)}\right|^p\right\} \leq E_{\mathbf{Q}}\left\{\frac{1}{U^{(\epsilon)}(t,x)^{2p}}\right\}^{1/2} \\
&\quad \times E_{\mathbf{Q}}\left\{E_{\mathbf{P}}\left[\exp\left\{-\frac{4p}{\epsilon}\left(\sum_{n\geq 1} a_n\left(\int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right)d\beta_s^{1n}\right.\right.\right.\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left.\left.\left.\left.+\int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right)d\beta_s^{2n}\right)\right)\right\}\right]\right\}^{1/4} \\
&\quad \times E_{\mathbf{Q}}\left\{E_{\mathbf{P}}\left[\left(\sum_{n\geq 1} n a_n\left(\int_0^t \sin\left(n(x+w_{t-s}^{(\epsilon)})\right)d\beta_s^{1n} - \int_0^t \cos\left(n(x+w_{t-s}^{(\epsilon)})\right)d\beta_s^{2n}\right)\right)^{4p}\right]\right\}^{1/4} \\
&\quad \leq \exp\left\{\frac{p^2}{\epsilon^2}t\left(\sum_{n\geq 1} a_n^2\right)\right\} \times \exp\left\{\frac{2p^2}{\epsilon^2}t\left(\sum_{n\geq 1} a_n^2\right)\right\} \times \left(\prod_{j=1}^{2p}(2j-1)\right)^{1/4} t^{p/2}\left(\sum_{n\geq} n^2 a_n^2\right)^{p/2} \\
&\quad < +\infty,
\end{aligned}
$$

thus establishing inequality (27).

$\square$

Theorem 7. *Recall the definition of $\mathcal{S}_p^*$, in equation (12) in the statement of lemma 5. For fixed $\epsilon > 0$ and each $0 < p < +\infty$, under hypothesis 1 that $\sum_{n\geq 1} n^4|a_n| < +\infty$, there is existence and uniqueness of solution to equation (1) in $\mathcal{S}_p^*$ for each $0 < p < +\infty$. This solution has a version such that for any $\epsilon > 0$ and for all $t \in [0,T]$, $u^{(\epsilon)}(t,.) \in C^{2,\gamma}([0,2\pi])$ for all $\gamma < 1$ and, for fixed $x \in \mathbf{R}$, $u^{(\epsilon)}(.,x)$ is Hölder continuous of all orders less than $1/2$.*

Proof Let $U^{(\epsilon)}$ denote the unique solution to equation (13) in $\mathcal{S}_p$. Set

$$
u^{(\epsilon)}(t,x) := -\epsilon\frac{\partial}{\partial x}\log U^{(\epsilon)}(t,x) = \frac{-\epsilon U_x^{(\epsilon)}}{U^{(\epsilon)}}. \tag{28}
$$

Then, by lemma 6, $u^{(\epsilon)}$ is well defined, belongs to $\mathcal{S}_p$ and satisfies the regularity properties listed above. It is straightforward to show that it solves equation (1).

For uniqueness, let $U^{(\epsilon)}$ denote the unique solution to equation (13) in $\mathcal{S}_p$ and let $U^{(\epsilon)}f$ denote any other solution to equation (13) that is adapted and twice differentiable in the space variables. This is necessary to ensure that $-\epsilon(\log(U^{(\epsilon)}f))_x$ has the spatial regularity to belong to $\mathcal{S}_p^*$. Then, for each $x \in [0,2\pi]$, $U^{(\epsilon)}(.,x)f(.,x)$ is a semimartingale and

$$
\begin{aligned}
\partial_t(U^{(\epsilon)}f) &= \frac{\epsilon}{2}U_{xx}^{(\epsilon)}f\,dt - \frac{1}{\epsilon}U^{(\epsilon)}f\circ\partial_t\zeta + U^{(\epsilon)}\partial_t f + d\langle U^{(\epsilon)}, f\rangle_t \\
&= \frac{\epsilon}{2}(U^{(\epsilon)}f)_{xx}dt - \epsilon U_x^{(\epsilon)}f_x dt - \frac{\epsilon}{2}U^{(\epsilon)}f_{xx}dt - \frac{1}{\epsilon}U^{(\epsilon)}f\circ\partial_t\zeta + U^{(\epsilon)}\partial_t f + d\langle U^{(\epsilon)}, f\rangle_t
\end{aligned}
$$

so that

$$0 = -\epsilon U_x^{(\epsilon)} f_x dt - \frac{\epsilon}{2} U^{(\epsilon)} f_{xx} dt + U^{(\epsilon)} \partial_t f + d\langle U^{(\epsilon)}, f \rangle_t.$$

It follows that f is differentiable in the time variable and hence, using $u^{(\epsilon)}(t,x) = \frac{-\epsilon U_x^{(\epsilon)}(t,x)}{U^{(\epsilon)}(t,x)}$, that f satisfies

$$\begin{cases} \frac{\partial}{\partial t} f(t,x) = \frac{\epsilon}{2} f_{xx} + u^{(\epsilon)} f_x \\ f(0,x) \equiv 1. \end{cases} \tag{29}$$

Theorem 11 shows that, almost surely, $\sup_{0<\epsilon\leq 1} \sup_{0\leq x\leq 2\pi} \sup_{0\leq t\leq T} |u^{(\epsilon)}(t,x)| < C(T)$ where $E_{\mathbf{Q}}\{C(T)^p\} < +\infty$ for each $p > 0$. It follows that, among functions that are 2π-periodic in the space variable, the unique solution to equation (29) is $f \equiv 1$.

Let $\tilde{u}^{(\epsilon)}(t,x)$ denote any adapted solution to equation (1). Set

$$\tilde{U}^{(\epsilon)}(t,x) = U^{(\epsilon)}(t,0) \exp\left\{ -\frac{1}{\epsilon} \int_0^x \tilde{u}^{(\epsilon)}(t,y) dy \right\}.$$

It follows that $\tilde{U}^{(\epsilon)}$ satisfies equation (13). Since equation (13) has a *unique* solution (by lemma 5), it follows (since $U^{(\epsilon)}$ is differentiable) that $\tilde{u}^{(\epsilon)}$ is uniquely determined and hence that there is a unique solution to equation (1) in $\mathcal{S}_p^*$. The regularity follows directly from equation (28), the lower bound given by lemma 6 and the regularity results of lemma 5. □

For $\epsilon > 0$, under hypothesis 1, there is a solution to equation (1) $u^{(\epsilon)}(t,.)$ that for each $t \in [0,T]$ satisfies $u^{(\epsilon)}(t,.) \in C^{2,\gamma}(\mathbf{R})$ for each $\gamma < 1$ (twice differentiable, second derivative Hölder continuous of orders γ for each $\gamma < 1$ and for each $x \in [0,2\pi]$ satisfies $u^{(\epsilon)}(.,x) \in C^{0,\alpha}([0,T])$ (Hölder continuous of order α) for all $\alpha < \frac{1}{2}$, $\mathbf{Q}$ - almost surely. This follows from the regularity of $U^{(\epsilon)}$, the identity $u^{(\epsilon)} = \frac{-\epsilon U_x^{(\epsilon)}}{U^{(\epsilon)}}$ and the lower bound on $U^{(\epsilon)}$ computed in lemma 6. For this solution, it follows that (1) may be rewritten as

$$\begin{cases} \partial_t u = \left(\frac{\epsilon}{2} u_{xx}^{(\epsilon)} - u^{(\epsilon)} u_x^{(\epsilon)} \right) dt + \partial_t \zeta_x \\ u_0 = 0. \end{cases} \tag{30}$$

Following the bounds established later in theorem 11, which completes the uniqueness argument of theorem 7, it will follow that this is the unique solution of equation (1) in the space $\mathcal{S}_p$ for $0 < p < +\infty$.

One of the main tools for analysing the equation is to consider the infinitesimal generator

$$\mathcal{L}^{(\epsilon)}(t,x) := \frac{\epsilon}{2} \frac{\partial^2}{\partial x^2} - u^{(\epsilon)}(t,x) \frac{\partial}{\partial x}. \tag{31}$$

Appling the notation introduced in equation (31) to equation (30), equation (1) may be written as

$$\begin{cases} \partial_t u = \mathcal{L}^{(\epsilon)} u dt + \partial_t \zeta_x \\ u_0 = 0. \end{cases}$$

The process generated by the infinitesimal generator $\mathcal{L}^{(\epsilon)}$ is given in definition 8 and will be used extensively in the sequel; an 'implicit' representation of the solution to equation (1) will be formulated in terms of the process generated by $\mathcal{L}$ (the superscript will be dropped from the notation when it is clearly implied).

Definition 8. *Let w denote a standard Wiener process, $w_0 = 0$, independent of the β^{jk} and let $\mathcal{F}_{s,t}$ denote the sigma algebra generated by the increments $(w_\alpha - w_\beta)_{s \leq \beta \leq \alpha \leq t}$. Let $\mathbf{P}$ denote the probability measure under which w is a standard Wiener process and let $E_{\mathbf{P}}$ denote expectation with respect to $\mathbf{P}$. Let $X^{(\epsilon)}$ denote the stochastic process defined as the unique solution to the stochastic integral equation*

$$X_{s,t}^{(\epsilon)}(x) = x + \sqrt{\epsilon}(w_t - w_s) - \int_s^t u^{(\epsilon)} \left(r, X_{r,t}^{(\epsilon)}(x) \right) dr. \tag{32}$$

From the regularity results on $u^{(\epsilon)}$ (globally Lipschitz in the space variable, Hölder continuous on the time variable and bounded in $[0,T] \times [0,2\pi]$), it follows directly from standard results that $\mathbf{Q}$-almost surely, $X^{(\epsilon)}$ is well defined, with pathwise uniqueness.

Definition 9. *Let S^1 denote $\mathbf{R}$ with the identification $x + 2\pi = x$ and let $C(S^1)$ denote continuous 2π periodic functions. The operator $\mathcal{Q}_{s,t} : C(S^1) \to C(S^1)$ is defined as*

$$\mathcal{Q}_{s,t} f(x) = E_{\mathbf{P}}[f(X_{s,t}(x))].$$

Note that, for $f \in C(S^1)$,

$$\frac{\partial}{\partial s} \mathcal{Q}_{s,t} f(x) = -\mathcal{Q}_{s,t}(\mathcal{L}_s f)(x) \tag{33}$$

and

$$\frac{\partial}{\partial t} \mathcal{Q}_{s,t} f(x) = \mathcal{L}_t(x) \mathcal{Q}_{s,t} f(x), \tag{34}$$

where $\mathcal{L}$ is defined by equation (31).

Lemma 10. *The following identity holds.*

$$\begin{aligned} u^{(\epsilon)}(t,x) = &- \sum_{n \geq 1} n a_n \left(\sin(nx) \beta^{1n}(t) - \cos(nx) \beta^{2n}(t) \right) \\ &- \sum_{n \geq 1} n a_n \left(\int_0^t \beta^{1n}(s) \mathcal{Q}_{s,t}(\mathcal{L}_s \sin(n.))(x) ds - \int_0^t \beta^{2n}(s) \mathcal{Q}_{s,t}(\mathcal{L}_s \cos(n.))(x) ds \right). \end{aligned} \tag{35}$$

Proof of lemma 10 For $s = s_0 < s_1 < \ldots < s_m = t$, using equation (33),

$$\begin{aligned}
u^{(\epsilon)}(t,x) &= u^{(\epsilon)}(t,x) - \mathcal{Q}_{0,t}u^{(\epsilon)}(0,x) \\
&= \sum_{k=0}^{m-1}(\mathcal{Q}_{s_{k+1},t}u(s_{k+1},x) - \mathcal{Q}_{s_{j_k},t}u(s_k,x)) \\
&= \sum_{k=0}^{m-1}\mathcal{Q}_{s_{k+1},t}(u(s_{k+1},x) - u(s_k,x)) + \sum_{k=0}^{m-1}(\mathcal{Q}_{s_{k+1},t} - \mathcal{Q}_{s_k,t})u(s_k,x) \\
&= \sum_{k=0}^{m-1}\mathcal{Q}_{s_{k+1},t}\int_{s_k}^{s_{k+1}}(\mathcal{L}_s u(s,.))(x)ds \\
&\quad + \sum_{k=0}^{m-1}\sum_{n\geq 1} na_n\left(\mathcal{Q}_{s_{k+1},t}\sin(nx)(\beta^{1n}_{s_{k+1}} - \beta^{1n}_{s_k}) - \mathcal{Q}_{s_{k+1},t}\cos(nx)(\beta^{2n}_{s_{k+1}} - \beta^{2n}_{s_k})\right) \\
&\quad - \sum_{k=0}^{m-1}\int_{s_k}^{s_{k+1}}\mathcal{Q}_{s,t}(\mathcal{L}_s u(s_k,.))(x)ds \\
&= \sum_{k=1}^{m}\int_{s_k}^{s_{k+1}}(\mathcal{Q}_{s_{k+1},t} - \mathcal{Q}_{s,t})(\mathcal{L}_s u(s_k,.))(x)ds \\
&\quad - \sum_{k=1}^{m-1}\sum_{n\geq 1} na_n\left(\beta^{1n}_{s_k}(\mathcal{Q}_{s_{k+1},t} - \mathcal{Q}_{s_k,t})\sin(nx) - \beta^{2n}_{s_k}(\mathcal{Q}_{s_{k+1},t} - \mathcal{Q}_{s_k,t})\cos(nx)\right) \\
&\quad + \sum_{n\geq 1} na_n\left(\beta^{1n}_t\sin(nx) - \beta^{2n}_t\cos(nx)\right).
\end{aligned}$$

This holds for any partition $0 = s_0 < \ldots < s_m = t$. Now, let the mesh size tend to zero. The convergence details are standard and give the advertised result. $\square$

The next theorem gives bounds on the moments of the solution.

Theorem 11. *Suppose that hypothesis 1 is satisfied. Let $u^{(\epsilon)}$ denote the solution to equation (1). Let $G(p)$ be the constant given in equation (9), defined in lemma 4. Then*

$$\begin{aligned}
&E_{\mathbf{Q}}\left\{\left(\sup_{0\leq\epsilon\leq 1}\sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}|u^{(\epsilon)}(s,x)|\right)^p\right\} \\
&\leq C_1(p,t)\left(\sum_{n\geq 1}n^3|a_n|\right)^p \exp\left\{C_2(p,t)\sum_{n\geq 1}n^6a_n^2 + C_3(p,t)\sum_{n\geq 1}n^3|a_n|\right\} = K(p,T), \quad (36)
\end{aligned}$$

where

$$C_1(p,t) = (2+t)^{3p/2}((2\log 2)^p + 4pG(2p-1))^{1/2},$$

$$C_2(p,t) = 2p^2t^3$$

and

$$C_3(p,t) = 2\sqrt{2}(2\sqrt{\pi} + \sqrt{\log 2})pt^{3/2}.$$

This result is consequence of the following lemma with $\tilde{b} = 1$, where $\tilde{b}$ is defined in the statement of the lemma.

Lemma 12. *Set*

$$f_{1b}(y) = \begin{cases} \cos(y) & b = 4n \\ -\sin(y) & b = 4n+1 \\ -\cos(y) & b = 4n+2 \\ \sin(y) & b = 4n+3 \end{cases}$$

for non negative integer n. Set $f_{2b}(y) = f_{1,(b+3)}(y)$. Set

$$\begin{aligned} \theta^{(\epsilon)}(b;t,x) &= \sum_{n\geq 1} n^b a_n \left(f_{1b}(nx)\beta_t^{1n} + f_{2b}(nx)\beta_t^{2n} \right) \\ &+ \sum_{n\geq 1} n^b a_n \left(\int_0^t \mathcal{Q}_{s,t}(\mathcal{L}_s f_{1b}(n.))(x)\beta_s^{1n} ds + \int_0^t \mathcal{Q}_{s,t}(\mathcal{L}_s f_{2b}(n.))(x)\beta_s^{2n} \right) ds. \end{aligned} \tag{37}$$

For $b_j \geq 1$, set $\tilde{b} = \sup(b_1, \ldots, b_p)$. Suppose that

$$\sum_{n=1}^{\infty} n^{2+\tilde{b}} |a_n| < +\infty,$$

so that

$$\sum_{n=1}^{\infty} n^{2(2+\tilde{b})} a_n^2 < +\infty.$$

Then

$$\begin{aligned} &E_{\mathbf{Q}} \left\{ \prod_{j=1}^{p} \left(\sup_{0\leq\epsilon\leq 1} \sup_{0\leq s\leq t} \sup_{0\leq x\leq 2\pi} \left| \theta^{(\epsilon)}(b_j, s, x) \right| \right) \right\} \\ &\leq C_1(p,t) \left(\sum_{n\geq 1} n^{2+\tilde{b}} |a_n| \right)^p \exp\left\{ C_2(p,t) \sum_n n^{2(2+\tilde{b})} a_n^2 + C_3(p,t) \sum_{n\geq 1} n^{2+\tilde{b}} |a_n| \right\}, \end{aligned} \tag{38}$$

where

$$C_1(p,t) = (2+t)^{3p/2} \left((2\log 2)^p + 4pG(2p-1) \right)^{1/2},$$

$$C_2(p,t) = 4p^2t^3$$

and

$$C_3(p,t) = 4\sqrt{2}(2\sqrt{\pi} + \sqrt{\log 2})pt^{3/2}.$$

Note that $u^{(\epsilon)}(t,x) = \theta^{(\epsilon)}(1;t,x)$.

Proof of lemma 12 Note that

$$\begin{aligned}\mathcal{Q}_{s,t}(\mathcal{L}_s \sin(n.))(x) &= \left(\mathcal{Q}_{s,t}\left(\frac{\epsilon}{2}\partial^2_{xx} - u^{(\epsilon)}(s,.)\partial_x\right)\sin(n.)\right)(x) \\ &= -nE_{\mathbf{P}}\left[u^{(\epsilon)}\left(s, X^{(\epsilon)}_{s,t}(x)\right)\cos\left(nX^{(\epsilon)}_{s,t}(x)\right)\right] - n^2\frac{\epsilon}{2}E_{\mathbf{P}}\left[\sin\left(nX^{(\epsilon)}_{s,t}(x)\right)\right]\end{aligned}$$

and, similarly,

$$\mathcal{Q}_{s,t}(\mathcal{L}_s \cos(n.))(x) = nE_{\mathbf{P}}\left[u^{(\epsilon)}\left(s, X^{(\epsilon)}_{s,t}(x)\right)\sin\left(nX^{(\epsilon)}_{s,t}(x)\right)\right] - n^2\frac{\epsilon}{2}E_{\mathbf{P}}\left[\cos\left(nX^{(\epsilon)}_{s,t}(x)\right)\right].$$

It follows that, for $j = 1, 2$,

$$\mathcal{Q}_{s,t}(\mathcal{L}_s f_{jb}(n.))(x) = -nE_{\mathbf{P}}\left[u^{(\epsilon)}\left(s, X^{(\epsilon)}_{s,t}(x)\right)f_{j,(b+1)}\left(nX^{(\epsilon)}_{s,t}(x)\right)\right] - n^2\frac{\epsilon}{2}E_{\mathbf{P}}\left[f_{jb}\left(nX^{(\epsilon)}_{s,t}(x)\right)\right].$$

Equation (37) therefore gives

$$\begin{aligned}\theta^{(\epsilon)}(b;t,x) &= \sum_{n\geq 1} n^b a_n\left(f_{1b}(nx)\beta^{1n}(t) + f_{2b}(nx)\beta^{2n}(t)\right) \\ &-\sum_{n\geq 1} n^{1+b}a_n\left(\int_0^t\left(\beta^{1n}(s)E_{\mathbf{P}}\left[u^{(\epsilon)}(s, X^{(\epsilon)}_{s,t}(x))f_{1(b+1)}(nX^{(\epsilon)}_{s,t}(x))\right]\right.\right. \\ &\left.\left.+\beta^{2n}(s)E_{\mathbf{P}}\left[u^{(\epsilon)}(s, X^{(\epsilon)}_{s,t}(x))f_{2,(b+1)}\left(nX^{(\epsilon)}_{s,t}(x)\right)\right]\right)ds\right) \\ &-\frac{\epsilon}{2}\sum_{n\geq 1} n^{2+b}a_n\left(\int_0^t\left(\beta^{1n}(s)E_{\mathbf{P}}\left[f_{1b}\left(nX^{(\epsilon)}_{s,t}(x)\right)\right] + \beta^{2n}(s)E_{\mathbf{P}}\left[f_{2b}\left(nX^{(\epsilon)}_{s,t}(x)\right)\right]\right)ds\right). \quad (39)\end{aligned}$$

Now set

$$\tilde{B}(b,t) = \sum_{n\geq 1} n^b|a_n|\left(\sup_{0\leq s\leq t}\left|\beta^{1n}(s)\right| + \sup_{0\leq s\leq t}\left|\beta^{2n}(s)\right|\right). \quad (40)$$

Set

$$\tilde{C}^{(\epsilon)}(b,t) = \sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|\theta^{(\epsilon)}(b,s,x)\right|. \quad (41)$$

In particular, from equation (37),

$$\tilde{C}^{(\epsilon)}(1,t) = \sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|u^{(\epsilon)}(s,x)\right|.$$

It follows from equation (39) that

$$\tilde{C}^{(\epsilon)}(b,t) \leq \tilde{B}(b,t) + \int_0^t \tilde{C}^{(\epsilon)}(1,s)\tilde{B}(b+1,s)ds + \frac{\epsilon t}{2}\tilde{B}(b+2,t). \tag{42}$$

Set

$$\tilde{C}(b,t) = \sup_{0<\epsilon<1}\tilde{C}^{(\epsilon)}(b,t), \tag{43}$$

so that

$$\tilde{C}(1,t) = \sup_{0<\epsilon\leq 1}\sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|u^{(\epsilon)}(s,x)\right|.$$

The extraordinary level of detail in the following very, very simple Gronwall argument has been inserted for the benefit of the mathematically challenged, some of whom amazingly seem to be employed on the editorial boards of prestigious journals.

Note that $\tilde{B}(b,t)$ defined by equation (40) is increasing as b increases for $b > 0$. This has to be pointed out, because it is apparently not self - evident. Let $D^{(\epsilon)}(1,t)$ denote the solution to

$$D^{(\epsilon)}(1,t) = (1+\frac{\epsilon t}{2})\tilde{B}(3,t) + \int_0^t D^{(\epsilon)}(1,s)\tilde{B}(3,s)ds,$$

so that (very, very clearly)

$$D^{(\epsilon)}(1,t) \leq (1+\frac{\epsilon t}{2})\tilde{B}(3,t)e^{\int_0^t \tilde{B}(3,s)ds}$$

and let $D^{(\epsilon)}(b,t)$ solve

$$D^{(\epsilon)}(b,t) = \tilde{B}(b+2,t) + \int_0^t D^{(\epsilon)}(1,s)\tilde{B}(b+2,s)ds + \frac{\epsilon t}{2}\tilde{B}(b+2,t)ds.$$

then it should be clear to anyone with a brain without further explanation that $\tilde{C}^{(\epsilon)}(b,t) \leq D^{(\epsilon)}(b,t)$ and, since $\tilde{B}(b,s)$ is increasing in b, it follows directly that $D^{(\epsilon)}(b,s) \geq D^{(\epsilon)}(1,s)$ for $b \geq 1$. It therefore follows that

$$D^{(\epsilon)}(b,t) \leq \tilde{B}(b+2,t) + \int_0^t D^{(\epsilon)}(b,s)\tilde{B}(b+2,s)ds + \frac{\epsilon t}{2}\tilde{B}(b+2,t).$$

Since $\tilde{B}(b,s)$ is increasing in s, it follows that for all $0 \leq s \leq t$ and $0 < \epsilon \leq 1$,

$$D^{(\epsilon)}(b,s) \le \left(1+\frac{t}{2}\right)\tilde{B}(b+2,t) + \tilde{B}(b+2,t)\int_0^s D^{(\epsilon)}(b,r)dr.$$

From this, it is it follows that, for $0 < \epsilon \le 1$,

$$\tilde{C}^{(\epsilon)}(b,t) \le D^{(\epsilon)}(b,t) \le \left(1+\frac{t}{2}\right)\tilde{B}(b+2,t)\exp\left\{t\tilde{B}(b+2,t)\right\}. \tag{44}$$

where $\tilde{C}^{(\epsilon)}(b,t)$ is defined by equation (41). Set

$$\tilde{D}(b,t) := \left(1+\frac{t}{2}\right)\tilde{B}(b+2,t)\exp\left\{t\tilde{B}(b+2,t)\right\}, \tag{45}$$

so that, from equation (44),

$$\sup_{0<\epsilon\le 1}\tilde{C}^{(\epsilon)}(b,t) \le \tilde{D}(b,t). \tag{46}$$

It follows, using $\tilde{b} = b_1 \vee \ldots \vee b_p$, that

$$E_{\mathbf{Q}}\left\{\prod_{j=1}^{p}\sup_{0<\epsilon<1}\sup_{0\le s\le t}\sup_{x}\left|\theta^{(\epsilon)}(b_j;s,x)\right|\right\} \le \left(1+\frac{t}{2}\right)^p E_{\mathbf{Q}}\left\{\tilde{B}(\tilde{b}+2,t)^p\exp\left\{pt\tilde{B}(\tilde{b}+2,t)\right\}\right\}.$$

Using the bounds calculated in lemmas 3 and 4, and recalling equation (40), it follows that

$$\begin{aligned}
&E_{\mathbf{Q}}\left\{\prod_{j=1}^{p}\sup_{0\le\epsilon\le 1}\sup_{0\le s\le t}\sup_{0\le x\le 2\pi}\left|\theta^{(\epsilon)}(b_j;s,x)\right|\right\}\\
&\le \left(1+\frac{t}{2}\right)^p E_{\mathbf{Q}}\left\{\tilde{B}(\tilde{b}+2,t)^{2p}\right\}^{1/2} E_{\mathbf{Q}}\left\{\exp\left\{2pt\tilde{B}(\tilde{b}+2,t)\right\}\right\}^{1/2}\\
&\le (2+t)^{3p/2}\left(\sum_n n^{2+\tilde{b}}|a_n|\right)^p\left((2\log 2)^p + 4pG(2p-1)\right)^{1/2}\\
&\quad\times\exp\left\{4p^2t^3\sum_n n^{2(2+\tilde{b})}a_n^2 + \left(2\sqrt{\pi}+\sqrt{\log 2}\right)4\sqrt{2}pt^{3/2}\sum_n n^{2+\tilde{b}}|a_n|\right\},
\end{aligned}$$

which is the bound advertised in the statement of lemma 12. □

Proof of Theorem 11 This follows directly from lemma 12 with $b_1 = \ldots = b_p = 1$. □

The next theorem ensures that the moment fields are uniformly Lipschitz in the space variables.

Theorem 13. *Let $u^{(\epsilon)}$ denote the solution to equation (1), under the condition that*

$$\sum_{n\geq 1} n^4|a_n| < +\infty,$$

so that

$$\sum_{n\geq 1} n^8 a_n^2 < +\infty.$$

It holds that

$$\sup_{0\leq s\leq t}\sup_{x_1,\ldots,x_p} E_{\mathbf{Q}}\left\{\left|u_x^{(\epsilon)}(s,x_1)u^{(\epsilon)}(s,x_2)\ldots u^{(\epsilon)}(s,x_p)\right|\right\} \leq K(p,t)$$

where $K(p,t)$ is independent of ϵ and is given by

$$K(p,t) = \left(C_1(p,t)\left(\sum_n n^4|a_n|\right)^p + C_2(p,t)\left(\sum_n n^4|a_n|\right)^{p+1}\right) \\ \times \exp\left\{C_3(p,t)\sum_n n^8 a_n^2 + C_4(p,t)\sum_n n^4|a_n|\right\},$$

where

$$C_1(p,t) = (2+t)^{3p/2}((2\log 2)^p + 4pG(2p-1))^{1/2},$$
$$C_2(p,t) = (2+t)^{3(p+1)/2}((2\log 2)^{p+1} + 4(p+1)G(2p+1))^{1/2},$$
$$C_3(p,t) = 2(p+1)^2 t^3$$

and

$$C_4(p,t) = 2(p+1)t^{3/2}(\sqrt{2}(\sqrt{\log 2} + 2\sqrt{\pi})).$$

Recall the definition of $m_p^{(\epsilon)}$ given in equation (3),

$$m_p^{(\epsilon)}(t;x_1,\ldots,x_p) := E_{\mathbf{Q}}\left\{\prod_{j=1}^p u^{(\epsilon)}(t,x_j)\right\}.$$

Then

$$\sup_{1\leq j\leq p}\sup_{0\leq s\leq t}\sup_{0<\epsilon\leq 1}\left|\frac{\partial}{\partial x_j} m_p^{(\epsilon)}(s;x_1,\ldots,x_p)\right| \leq K(p,t),$$

where $K(p,t)$ is described above.

This theorem is a consequence of the following lemma.

Lemma 14. *Using the notations of lemma 12, recall that*

$$\tilde{C}(b,t) := \sup_{0\le s\le t}\sup_{0\le\epsilon\le 1}\sup_{0\le x\le 2\pi}\left|\theta^{(\epsilon)}(b;s,x)\right|.$$

With change of notation, set $\tilde{b} = 2\vee\max(b_1,\dots,b_{p-1})$ *(the change is the* 2*) and suppose that* $(a_n)_{n\ge 1}$ *satisfies*

$$\sum_{n\ge 1} n^{2+\tilde{b}}|a_n| < +\infty,$$

so that

$$\sum_{n\ge 1} n^{2(2+\tilde{b})}a_n^2 < +\infty.$$

Then, for $0\le t<+\infty$,

$$\sup_{0\le s\le t} E_{\mathbf{Q}}\left\{\tilde{C}(b_1,t)\dots\tilde{C}(b_{p-1},t)\left|u_x^{(\epsilon)}(s,x)\right|\right\} \le K(p;\tilde{b},t),$$

where

$$K(p;\tilde{b},t) := \left(C_1(p,t)\left(\sum_n n^{2+\tilde{b}}|a_n|\right)^p + C_2(p,t)\left(\sum_n n^{2+\tilde{b}}|a_n|\right)^{p+1}\right) \times\exp\left\{C_3(p,t)\sum_n n^{4+2\tilde{b}}a_n^2 + C_4(p,t)\sum_n n^{2+\tilde{b}}|a_n|\right\}, \tag{47}$$

where

$$C_1(p,t) = (2+t)^{3p/2}((2\log 2)^p + 4pG(2p-1))^{1/2},$$
$$C_2(p,t) = (2+t)^{3(p+1)/2}((2\log 2)^{p+1} + 4(p+1)G(2p+1))^{1/2},$$
$$C_3(p,t) = 2(p+1)^2t^3$$

and

$$C_4(p,t) = 2(p+1)t^{3/2}(\sqrt{2}(\sqrt{\log 2}+2\sqrt{\pi})).$$

Proof of lemma 14 Recall the definition of X given in equation (32). Let $\mathcal{F}_{s,t}$ denote the σ-algebra generated by the increments $w_v - w_u,\ s\le u<v\le t$. From equation (35), note that

$$
\begin{aligned}
u^{(\epsilon)}\left(s, X^{(\epsilon)}_{s,t}(x)\right) &= -\sum_{n\geq 1} n a_n \left(\sin\left(nX^{(\epsilon)}_{s,t}(x)\right)\beta^{1n}(s) - \cos\left(nX^{(\epsilon)}_{s,t}(x)\right)\beta^{2n}(s)\right) \\
&+\sum_{n\geq 1} n^2 a_n \left(\int_0^s \left(\beta^{1n}(r) E_{\mathbf{P}}\left[u\left(r, X^{(\epsilon)}_{r,t}\right)\cos\left(nX^{(\epsilon)}_{r,t}(x)\right)\middle|\mathcal{F}_{s,t}\right]\right.\right. \\
&\qquad\left.\left. -\beta^{2n}(r) E_{\mathbf{P}}\left[u\left(r, X^{(\epsilon)}_{r,t}(x)\right)\sin\left(nX^{(\epsilon)}_{r,t}(x)\right)\middle|\mathcal{F}_{s,t}\right]\right) dr\right) \\
&+\frac{\epsilon}{2}\sum_{n\geq 1} n^3 a_n \left(\int_0^s \left(\beta^{1n}(r) E_{\mathbf{P}}\left[\sin\left(nX^{(\epsilon)}_{r,t}(x)\right)\middle|\mathcal{F}_{s,t}\right] - \beta^{2n}(r) E_{\mathbf{P}}\left[\cos\left(nX^{(\epsilon)}_{r,t}(x)\right)\middle|\mathcal{F}_{s,t}\right]\right) dr\right).
\end{aligned}
$$

Taking derivative in x, and using X' to denote X differentiated with respect to x gives

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(u^{(\epsilon)}(s, X^{(\epsilon)}_{s,t}(x))\right) \\
&\quad= -X^{(\epsilon)\prime}_{s,t}(x)\sum_{n\geq 1} n^2 a_n \left(\cos\left(nX^{(\epsilon)}_{s,t}(x)\right)\beta^{1n}(s) + \sin\left(nX^{(\epsilon)}_{s,t}(x)\right)\beta^{2n}(s)\right) \\
&\quad -E_{\mathbf{P}}\left[\int_0^s X^{(\epsilon)\prime}_{r,t}(x)\sum_{n\geq 1} n^3 a_n \left(\beta^{1n}(r) u^{(\epsilon)}\left(r, X^{(\epsilon)}_{r,t}\right)\sin\left(nX^{(\epsilon)}_{r,t}(x)\right)\right.\right. \\
&\qquad\qquad\left.\left. +\beta^{2n}(r) u^{(\epsilon)}\left(r, X^{(\epsilon)}_{r,t}(x)\right)\cos\left(nX^{(\epsilon)}_{r,t}(x)\right)\right) dr\middle|\mathcal{F}_{s,t}\right] \\
&\quad +E_{\mathbf{P}}\left[\int_0^s \frac{\partial}{\partial x}\left(u^{(\epsilon)}(r, X^{(\epsilon)}_{r,t}(x))\right)\sum_{n\geq 1} n^2 a_n \left(\beta^{1n}(r)\cos\left(nX^{(\epsilon)}_{r,t}(x)\right)\right.\right. \\
&\qquad\qquad\left.\left. -\beta^{2n}(r)\sin\left(nX^{(\epsilon)}_{r,t}(x)\right)\right) dr\middle|\mathcal{F}_{s,t}\right] \\
&\quad +\frac{\epsilon}{2} E_{\mathbf{P}}\left[\int_0^s X^{(\epsilon)\prime}_{r,t}(x)\sum_{n\geq 1} n^4 a_n \left(\beta^{1n}(r)\cos\left(nX^{(\epsilon)}_{r,t}(x)\right) + \beta^{2n}(r)\sin\left(nX^{(\epsilon)}_{r,t}(x)\right)\right) dr\middle|\mathcal{F}_{s,t}\right].
\end{aligned}
$$

Now, set

$$
g^{(t)}(s) = \frac{1}{2\pi}\int_0^{2\pi} E_{\mathbf{P}}\left[\left|\frac{\partial}{\partial x}\left(u^{(\epsilon)}(s, X^{(\epsilon)}_{s,t}(x))\right)\right|\right] dx.
$$

and recall the notation

$$
\tilde{C}(1,t) = \sup_{0<\epsilon<1}\sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|u^{(\epsilon)}(s,x)\right|.
$$

Recall the definition of $\tilde{B}(b,t)$ given in equation (40).

Note that $X^{(\epsilon)\prime}_{s,t}(x) \geq 0$ and that $\frac{1}{2\pi}\int_0^{2\pi} X^{(\epsilon)\prime}_{s,t}(x) dx = 1$. The above analysis yields

$$
g^{(t)}(s) \leq \tilde{B}(2,s) + s\tilde{C}(1,s)\tilde{B}(3,s) + \frac{\epsilon}{2} s\tilde{B}(4,s) + \tilde{B}(2,s)\int_0^s g^{(t)}(r) dr.
$$

Using the fact that $\tilde{B}(b,s)$ is increasing in b yields

$$g^{(t)}(s) \leq \tilde{B}(4,s) + s\left(\tilde{C}(1,s) + \frac{\epsilon}{2}\right)\tilde{B}(4,s) + \tilde{B}(4,s)\int_0^s g^{(t)}(r)dr.$$

Since $\tilde{B}(b,s)$ and $\tilde{C}(1,s)$ are increasing in s, it follows that for any $0 \leq r \leq s \leq t$ and any $0 < \epsilon \leq 1$

$$g^{(t)}(r) \leq \tilde{B}(4,s) + s\left(\tilde{C}(1,s) + \frac{1}{2}\right)\tilde{B}(4,s) + \tilde{B}(4,s)\int_0^r g^{(t)}(\alpha)d\alpha,$$

yielding that for any $0 < \epsilon \leq 1$ and any $s \leq t$,

$$g^{(t)}(s) \leq \tilde{B}(4,s)\left(1 + s\left(\tilde{C}(1,s) + \frac{1}{2}\right)\right)\exp\left\{s\tilde{B}(4,s)\right\}.$$

Recall equations (42) and (45), which give

$$\tilde{C}(1,s) \leq \left(1 + \frac{s}{2}\right)\tilde{B}(3,s)\exp\left\{s\tilde{B}(3,s)\right\},$$

from which (using $\tilde{B}(3,s) \leq \tilde{B}(4,s)$)

$$g^{(t)}(s) \leq \left(1 + \frac{s}{2}\right)\left(1 + s\tilde{B}(4,s)e^{s\tilde{B}(4,s)}\right)\tilde{B}(4,s)e^{s\tilde{B}(4,s)}.$$

Now, using $\tilde{b} = 2 \vee \max_{1 \leq j \leq p-1} b_j$, recall the definition of $\tilde{D}$ given in equation (45) and the inequality given in equation (46), for $0 \leq s \leq t < +\infty$, it follows by an application of lemmas 3 and 4 to get from the second last to the last line, that

$$\begin{aligned}
E_{\mathbf{Q}}&\left\{\tilde{C}(b_1,t)\ldots\tilde{C}(b_{p-1},t)\left|\frac{\partial u^{(\epsilon)}}{\partial x}(s,x)\right|\right\} \leq E_{\mathbf{Q}}\left\{\tilde{D}^{p-1}(\tilde{b},t)\left|\frac{\partial u^{(\epsilon)}}{\partial x}(s,x)\right|\right\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} E_{\mathbf{Q}}\left\{\tilde{D}^{p-1}(\tilde{b},t)\left|\frac{\partial u^{(\epsilon)}}{\partial x}(s,x)\right|\right\}dx = E_{\mathbf{Q}}\left\{\tilde{D}^{p-1}(\tilde{b},t)g^{(s)}(s)\right\} \\
&\leq \left(1 + \frac{t}{2}\right)^p E_{\mathbf{Q}}\left\{\tilde{B}^p(\tilde{b}+2,t)e^{pt\tilde{B}(\tilde{b}+2,t)} + t\tilde{B}^{p+1}(\tilde{b}+2,t)e^{(p+1)t\tilde{B}(\tilde{b}+2,t)}\right\} \\
&\leq \left(1 + \frac{t}{2}\right)^p \left(E_{\mathbf{Q}}\left\{\tilde{B}^{2p}(\tilde{b}+2,t)\right\}^{1/2} E_{\mathbf{Q}}\left\{e^{2pt\tilde{B}(\tilde{b}+2,t)}\right\}^{1/2}\right. \\
&\qquad\qquad \left. + tE_{\mathbf{Q}}\left\{\tilde{B}^{2(p+1)}(\tilde{b}+2,t)\right\}^{1/2} E_{\mathbf{Q}}\left\{e^{2(p+1)t\tilde{B}(\tilde{b}+2,t)}\right\}^{1/2}\right) \\
\leq\ & K(p;\tilde{b},t),
\end{aligned}$$

where $K(p;\tilde{b},t)$ is the constant given in equation (47). The conclusion of the last line from the

second last follows by an application of lemmas 3 and 4. Lemma 4 gives

$$\begin{aligned} E_{\mathbf{Q}}\left\{\tilde{B}^{2p}(b,t)\right\} &= E_{\mathbf{Q}}\left\{\left(\sum_{n\geq 1} n^b |a_n| (S^{1n}(t) + S^{2n}(t))\right)^{2p}\right\} \\ &\leq \left(\sum_{n\geq 1} n^b |a_n|\right)^{2p} 2^{2p} t^p \left((2\log 2)^p + 4pG(2p-1)\right) \end{aligned}$$

and lemma 3 gives

$$E_{\mathbf{Q}}\left\{e^{2pt\tilde{B}(b,t)}\right\} \leq e^{4p^2t^3\sum_{n\geq 1} n^{2b}a_n^2 + 4pt^{3/2}\sqrt{2}(\sqrt{\log 2}+2\sqrt{\pi})}.$$

These bounds may be applied to give the desired result. $\square$

Proof of Theorem 13 Note that

$$\begin{aligned} \sup_{0\leq s\leq t}\left|\frac{\partial}{\partial x_j} m^{(\epsilon)}(s; x_1, \ldots, x_p)\right| &= \sup_{0\leq s\leq t}\left|E_{\mathbf{Q}}\left\{\left(\prod_{k\neq j} u^{(\epsilon)}(s, x_k)\right) u_x(s, x_j)\right\}\right| \\ &\leq \sup_{0\leq s\leq t} E_{\mathbf{Q}}\{\tilde{C}(1,s)^{p-1}|u_x(s,x_j)|\} \end{aligned}$$

and the result now follows by applying lemma 14 with $b_1 = \ldots = b_{p-1} = 1$. $\square$

Lemma 15. *Suppose that $\sum_{n\geq 1} n^4|a_n| < +\infty$ so that $\sum_{n\geq 1} n^8 a_n^2 < +\infty$. For any $T \geq 0$, there exists a constant $C(p,T)$ such that*

$$\sup_{0\leq t\leq T}\sup_{0\leq\epsilon\leq 1}\frac{\partial}{\partial\epsilon} m_{2p}^{(\epsilon)}(t, \mathbf{0}) \leq C(p,T).$$

Note that this lemma gives no information on a lower bound.

Proof For $\epsilon > 0$, set $\tilde{u} = \frac{\partial u^{(\epsilon)}}{\partial \epsilon}$. Then $\tilde{u}$ is differentiable in t and satisfies

$$\begin{cases} \tilde{u}_t = \frac{\epsilon}{2}\tilde{u}_{xx} - u\tilde{u}_x - u_x\tilde{u} + \frac{1}{2}u_{xx} \\ \tilde{u}(0,x) \equiv 0. \end{cases} \tag{48}$$

Recall (suppressing the notation ϵ from the process X) that

$$X_{s,t}(x) = x + \sqrt{\epsilon}(w_t - w_s) - \int_s^t u^{(\epsilon)}(r, X_{r,t}(x))\,dr,$$

so that

$$\frac{\partial X_{s,t}(x)}{\partial x} = 1 - \int_s^t u_x^{(\epsilon)}(r, X_{r,t}(x))\frac{\partial X_{r,t}(x)}{\partial x}dr$$

yielding

$$\frac{\partial X_{s,t}(x)}{\partial x} = e^{-\int_s^t u_x(r,X_{r,t}(x))dr}. \tag{49}$$

It follows from equation (48), using equation (49) to go from the first line to the second, that

$$\begin{aligned}
\tilde{u}(t,x) &= \frac{1}{2}\int_0^t E_{\mathbf{P}}\left[u_{xx}(s,X_{s,t}(x))e^{-\int_s^t u_x(r,X_{r,t}(x))dr}\right]ds \\
&= \frac{1}{2}\int_0^t E_{\mathbf{P}}\left[u_{xx}(s,X_{s,t}(x))\frac{\partial X_{s,t}(x)}{\partial x}\right]ds \\
&= \frac{1}{2}\frac{\partial}{\partial x}\int_0^t E_{\mathbf{P}}\left[u_x(s,X_{s,t}(x))\right]ds,
\end{aligned}$$

yielding (for $\epsilon > 0$)

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}m^{(\epsilon)}(t;x_1,\ldots,x_p) &= \sum_{j=1}^p E_{\mathbf{Q}}\left\{\left(\prod_{k\neq j}u(t,x_k)\right)\tilde{u}(t,x_j)\right\} \\
&= \frac{1}{2}\sum_{j=1}^p \frac{\partial}{\partial x_j}\int_0^t E_{\mathbf{Q}}\left\{\left(\prod_{k\neq j}u(t,x_k)\right)E_{\mathbf{P}}\left[u_x(s,X_{s,t}(x_j))\right]\right\}ds
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}m_p^{(\epsilon)}(t,\mathbf{0}) &= \frac{p}{2}\int_0^t \frac{1}{2\pi}\int_0^{2\pi} E_{\mathbf{Q}}\left\{u^{p-1}(t,x)\frac{\partial}{\partial x}E_{\mathbf{P}}\left[u_x(s,X_{s,t}(x))\right]\right\}dxds \\
&= -\frac{p(p-1)}{2}\int_0^t E_{\mathbf{Q}}\left\{u^{p-2}(t,x)u_x(t,x)E_{\mathbf{P}}\left[u_x(r,X_{r,t}(x))\right]\right\}dr. \qquad (50)
\end{aligned}$$

Now, set $v = u_x$ and note that

$$\begin{cases} \partial_t v = (\frac{\epsilon}{2}v_{xx} - v^2 - uv_x)dt + \partial_t \zeta_{xx} \\ v(0,x) \equiv 0. \end{cases}$$

It follows that

$$\begin{aligned}
v(t,x) &= -\sum_{n\geq 1} n^2 a_n \left(\beta^{1n}(t)\cos(nx) + \beta^{2n}(t)\sin(nx)\right) \\
&\quad -\sum_{n\geq 1} n^2 a_n \left(\int_0^t \beta^{1n}(s)\mathcal{Q}_{s,t}(\mathcal{L}_s\cos(n.))(x)ds + \int_0^t \beta^{2n}(s)\mathcal{Q}_{s,t}(\mathcal{L}_s\sin(n.))(x)ds\right) \\
&\quad -\int_0^t E_{\mathbf{P}}[v^2(s,X_{s,t})]ds.
\end{aligned}$$

Recall the definition of θ given in equation (37). For $b = 2$,

$$\begin{aligned}
\theta(2;t,x) &= -\sum_{n\geq 1} n^2 a_n \left(\beta^{1n}(t)\cos(nx) + \beta^{2n}(t)\sin(nx)\right) \\
&\quad -\sum_{n\geq 1} n^2 a_n \left(\int_0^t \beta^{1n}(s)\mathcal{Q}_{s,t}(\mathcal{L}_s\cos(n.))(x)ds + \int_0^t \beta^{2n}(s)\mathcal{Q}_{s,t}(\mathcal{L}_s\sin(n.))(x)ds\right)
\end{aligned}$$

so that

$$u_x(t,x) = \theta(2;t,x) - \int_0^t E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds.$$

Note that

$$E_{\mathbf{P}}[u_x(r,X_{r,t}(x))] = E_{\mathbf{P}}[\theta(2;r,X_{r,t}(x))] - \int_0^r E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds$$

It follows that

$$\begin{aligned}
&u_x(t,x)E_{\mathbf{P}}[u_x(r,X_{r,t}(x))] = \theta(2;t,x)E[\theta(2;r,X_{r,t})] \\
&\quad + \int_0^t E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds \int_0^r E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds \\
&\quad -\theta(2;t,x)\int_0^r E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds - E_{\mathbf{P}}[\theta(2;r,X_{r,t}(x))]\int_0^t E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds.
\end{aligned}$$

Putting this into equation (50) gives

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}m_{2p}^{(\epsilon)}(t,\mathbf{0}) &\leq -p(2p-1)\left(\int_0^t E_{\mathbf{Q}}\left\{u^{2(p-1)}(t,x)\theta(2;t,x)E_{\mathbf{P}}[\theta(2;r,X_{r,t}(x))]\right\}dr\right. \\
&\quad -\int_0^t E_{\mathbf{Q}}\left\{u^{2(p-1)}(t,x)\theta(2;t,x)\int_0^r E_{\mathbf{P}}[v^2(s,X_{s,t}(x)]ds\right\}dr \\
&\quad \left.-\int_0^t E_{\mathbf{Q}}\left\{u^{2(p-1)}(t,x)E_{\mathbf{P}}[\theta(2;r,X_{r,t}(x))]\int_0^t E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds\right\}dr\right).
\end{aligned}$$

Note that

$$0 \leq \int_0^r E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds \leq \int_0^t E_{\mathbf{P}}[v^2(s,X_{s,t}(x))]ds = \theta(2;t,x) - u_x(t,x),$$

from which it follows that

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}m_{2p}^{(\epsilon)}(t,\mathbf{0}) &\leq -p(2p-1)\left(\int_0^t E_{\mathbf{Q}}\left\{u^{2(p-1)}(t,x)\theta(2;t,x)E_{\mathbf{P}}[\theta(2;r,X_{r,t}(x))]\right\}dr\right. \\
&\quad -tE_{\mathbf{Q}}\left\{u^{2(p-1)}(t,x)\theta(2;t,x)(\theta(2;t,x) - u_x(t,x))\right\}dr \\
&\quad \left.-\int_0^t E_{\mathbf{Q}}\left\{u^{2(p-1)}(t,x)E_{\mathbf{P}}[\theta(2;r,X_{r,t}(x))](\theta(2;t,x) - u_x(t,x))\right\}dr\right).
\end{aligned}$$

Recall the definition of $\tilde{C}^{(\epsilon)}$ given in equation (41); namely,

$$\tilde{C}^{(\epsilon)}(b,t) = \sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|\theta^{(\epsilon)}(b,s,x)\right|.$$

In particular,

$$\tilde{C}^{(\epsilon)}(1,t) = \sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|u^{(\epsilon)}(s,x)\right|$$

and

$$\tilde{C}^{(\epsilon)}(2,t) = \sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\left|\theta^{(\epsilon)}(2;s,x)\right|.$$

It follows that

$$\begin{aligned}\sup_{0\leq s\leq t}\frac{\partial}{\partial\epsilon}m_{2p}^{(\epsilon)}(s,\mathbf{0}) &\leq p(2p-1)\\ &\times\left(3tE_{\mathbf{Q}}\left\{\tilde{C}^{(\epsilon)}(1,t)^{2(p-1)}\tilde{C}^{(\epsilon)}(2,t)^2\right\} + 2t\sup_{0\leq s\leq t}E_{\mathbf{Q}}\left\{\tilde{C}^{(\epsilon)}(1,t)^{2(p-1)}\tilde{C}^{(\epsilon)}(2,t)\left|u_x(s,x)\right|\right\}\right).\end{aligned}$$

The first term on the right hand side is bounded, independently of ϵ, by an application of lemma 12. The second term is bounded, independently of ϵ by an application of lemma 14. □

3 The Moment Equations

Having constructed a priori bounds for the moments of solutions to equation (1) and a priori bounds on the Lipschitz constant, which are independent of ϵ, the system of equations for the moment fields is now considered.

Recall that $u^{(\epsilon)}(t,.) \in C^{2,1}([0,2\pi])$. It follows from equation (1) that $u^{(\epsilon)}(.,x)$ is a semimartingale for each $x \in [0,2\pi]$ and therefore Itô's formula may be applied to $f(u^{(\epsilon)}(.,x_1),\ldots,u^{(\epsilon)}(.,x_p)) = u^{(\epsilon)}(.,x_1)\ldots u^{(\epsilon)}(.,x_p)$. Set $\Gamma(z) = \sum_{n\geq 1} a_n^2\cos(nz)$. Itô's formula yields

$$\begin{aligned}\prod_{j=1}^p u^{(\epsilon)}(t,x_j) &= \frac{\epsilon}{2}\int_0^t\sum_{k=1}^p u_{xx}^{(\epsilon)}(s,x_k)\prod_{j\neq k}u^{(\epsilon)}(s,x_j)ds - \int_0^t\sum_{k=1}^p u_x^{(\epsilon)}(s,x_k)\prod_{j=1}^p u^{(\epsilon)}(s,x_j)ds\\ &+\sum_{k=1}^p\int_0^t\left(\prod_{j\neq k}u^{(\epsilon)}(s,x_j)\right)\partial_s\zeta(s,x_k) + \sum_{j<k}\int_0^t\left(\prod_{l\neq j,k}u^{(\epsilon)}(s,x_l)\right)(-\Gamma''(x_j-x_k))ds. \quad (51)\end{aligned}$$

Recall the definition of $m_p^{(\epsilon)}$ given in equation (3). To obtain an equation for m_p from equation (51), it is necessary to show that Fubini's theorem may be used on each term of the right hand

side of equation (51) and that the martingale term is indeed a martingale. Fubini's theorem may be applied to the second and fourth terms, using theorems 13 and 11 respectively. For the (local) martingale term, note that by the Burkholder Davis Gundry inequality (theorem 4.1 and corollary 4.2 on pages 160 and 161 of Revuz and Yor [4]) there exist constants $K(q) < +\infty$ such that

$$\begin{aligned} & E_{\mathbf{Q}}\left\{\sup_{0\leq s\leq t}\left|\left(\int_0^s \prod_{j\neq k} u^{(\epsilon)}(r,x_j)\partial_r\zeta(r,x_j)\right)\right|^q\right\} \\ & \leq K(q)E_{\mathbf{Q}}\left\{\left(\int_0^t (\prod_{j\neq k} u^{(\epsilon)}(r,x_j))^2(-\Gamma''(0))dr\right)^{q/2}\right\} \\ & \leq K(q)(-\Gamma''(0))^{q/2}t^{(q/2)-1}\int_0^t E_{\mathbf{Q}}\left\{|u(r,x)|^q\right\}dr \end{aligned}$$

which is bounded above by theorem 11. This gives that the (local) martingale is uniformly integrable and is therefore a martingale.

The only 'problem' term is the first one on the right hand side. Recall equation (28), where $U^{(\epsilon)}$ is given by equation (13). This may be rewritten as

$$u^{(\epsilon)} = -\epsilon\frac{U_x^{(\epsilon)}}{U^{(\epsilon)}}.$$

Then

$$u_{xx}^{(\epsilon)} = -\epsilon\left(\frac{U_{xxx}^{(\epsilon)}}{U^{(\epsilon)}} - \frac{3U_{xx}^{(\epsilon)}U_x^{(\epsilon)}}{U^{(\epsilon)2}} + 2\frac{U_x^{(\epsilon)3}}{U^{(\epsilon)3}}\right). \tag{52}$$

The solution of equation (13), with initial condition $U^{(\epsilon)}(0,x) \equiv 1$, may be expressed using the Feynman Kacs representation in equation (24). Using

$$f(x) = \sum_{n\geq 1} a_n\left(\int_0^t \cos(n(x+w_{t-s}^{(\epsilon)}))d\beta_s^{1n} + \int_0^t \sin(n(x+w_{t-s}^{(\epsilon)}))d\beta_s^{2n}\right)$$

to simplify notation, where under $\mathbf{P}$, $w^{(\epsilon)}$ is a Brownian motion, with $w_0^{(\epsilon)} = 0$ and diffusion coefficient ϵ, it follows that

$$U_x^{(\epsilon)} = -\frac{1}{\epsilon}E_{\mathbf{P}}[e^{-\frac{1}{\epsilon}f(x)}f_x(x)], \tag{53}$$

$$U_{xx}^{(\epsilon)} = \frac{1}{\epsilon^2}E_{\mathbf{P}}[e^{-\frac{1}{\epsilon}f(x)}(f_x(x))^2] - \frac{1}{\epsilon}E_{\mathbf{P}}[e^{-\frac{1}{\epsilon}f(x)}f_{xx}(x)] \tag{54}$$

and

$$U^{(\epsilon)}_{xxx} = -\frac{1}{\epsilon^3}E_{\mathbf{P}}[e^{-\frac{1}{\epsilon}f(x)}(f_x(x))^3] + \frac{3}{\epsilon^2}E_{\mathbf{P}}[e^{-\frac{1}{\epsilon}f(x)}f_x(x)f_{xx}(x)] - \frac{1}{\epsilon}E_{\mathbf{P}}[e^{-\frac{1}{\epsilon}f(x)}f_{xxx}(x)]. \qquad (55)$$

To show that Fubini's theorem may be applied to the first term on the right hand side of equation (51), note that

$$E_{\mathbf{Q}}\left\{\left|u^{(\epsilon)}_{xx}(s,x_k)\right|\prod_{j\neq k}\left|u^{(\epsilon)}(s,x_j)\right|\right\} \qquad (56)$$

$$\leq E_{\mathbf{Q}}\left\{\left|u^{(\epsilon)}_{xx}(s,x)\right|^2\right\}^{1/2} E_{\mathbf{Q}}\left\{\left|u^{(\epsilon)}(s,x)\right|^{2(p-1)}\right\}^{1/2} \leq C(s)E_{\mathbf{Q}}\left\{\left|u^{(\epsilon)}_{xx}(s,x)\right|^2\right\}^{1/2},$$

where the constant $C(s) < +\infty$, increasing in s, is obtained by an application of theorem 11 and is independent of ϵ.

Since the arguments for all the terms obtained by applying equations (53), (54) and (55) to equation (52) in estimating the right hand side of equation (56) are similar, only one will be sketched. Note that (for example)

$$E_{\mathbf{Q}}\left\{\left|\frac{U^{(\epsilon)}_{xxx}}{U^{(\epsilon)}}\right|^2\right\} \leq E_{\mathbf{Q}}\left\{U^{(\epsilon)4}_{xxx}\right\}^{1/2} E_{\mathbf{Q}}\left\{\frac{1}{U^{(\epsilon)4}}\right\}^{1/2}.$$

Note that, since $f(x)$ is Gaussian, for any $q \in \mathbf{R}$,

$$E_{\mathbf{Q}}\left\{\left(e^{-\frac{1}{\epsilon}f(x)}\right)^q\right\} = e^{\frac{q^2}{2\epsilon^2}\Gamma(0)t}$$

It follows (using Jensen's inequality on the function $\frac{1}{x}$ for $x \in (0,+\infty)$, which is convex in that region) that

$$E_{\mathbf{Q}}\left\{\frac{1}{U^{(\epsilon)4}}\right\} = E_{\mathbf{Q}}\left\{\frac{1}{E_{\mathbf{P}}\left[e^{-\frac{1}{\epsilon}f(x)}\right]^4}\right\} \leq E_{\mathbf{Q}}\left\{E_{\mathbf{P}}\left[e^{\frac{4}{\epsilon}f(x)}\right]\right\} = e^{\frac{16}{\epsilon^2}\Gamma(0)t}.$$

Let $f^{(n)} = \frac{\partial^n}{\partial x^n}f(x)$. Note that $\mathbf{P}$ almost surely $f^{(n)}(x)$, is Gaussian with respect to $\mathbf{Q}$, with $E_{\mathbf{Q}}\left\{f^{(n)2q-1}\right\} = 0$ and $E_{\mathbf{Q}}\left\{f^{(n)2q}\right\} = (-1)^n(\Gamma^{(2n)}(0))^q t^q \prod_{j=1}^q (2j-1)$, for all integer $q \geq 1$, where $\Gamma^{(2n)}$ denotes the $2n$th derivative of Γ. It is now easy to use Hölder's inequality to compute an upper bound for $E_{\mathbf{Q}}\left\{U^{(\epsilon)2k}_{xxx}\right\}$, $E_{\mathbf{Q}}\left\{U^{(\epsilon)2k}_{xx}\right\}$ and $E_{\mathbf{Q}}\left\{U^{(\epsilon)2k}_{x}\right\}$ for all $k \geq 1$, since these will involve $\Gamma^{(2n)}(0)$ for $n \leq 3$, which is bounded by hypothesis 1. These bounds depend on ϵ and are increasing as $t \to +\infty$ and as $\epsilon \to 0$. It now follows that there exists a non negative function $C(\epsilon,t)$, increasing in t, such that for any $t < +\infty$ and any $\epsilon > 0$, $C(\epsilon,t) < +\infty$ and such that

$$\sup_{0\leq s\leq t}\sup_{(x_1,\ldots,x_p)\in\mathbf{R}^p} E_{\mathbf{Q}}\left\{\left|u^{(\epsilon)}_{xx}(s,x_k)\right|\prod_{j\neq k}\left|u^{(\epsilon)}(s,x_j)\right|\right\} \leq C(\epsilon,t).$$

It follows that, for fixed $\epsilon > 0$ and $t < +\infty$, Fubini's theorem may be applied to equation (51). It has already been seen that the martingale term is a martingale, starting from 0 at $t = 0$ and therefore has expected value 0. It follows that

$$\frac{\partial}{\partial t} m_p^{(\epsilon)}(t; x_1, \ldots, x_p) = \frac{\epsilon}{2} \Delta_{\mathbf{x}} m_p^{(\epsilon)}(t; x_1, \ldots, x_p) \tag{57}$$
$$-\frac{1}{2} \sum_{j=1}^{p} \frac{\partial}{\partial x_j} m_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) + \sum_{k<l} (-\Gamma''(x_k - x_l)) m_{p-2}^{(\epsilon)}(t; \hat{x_k}, \hat{x_l}),$$

$$m_p^{(\epsilon)}(0; x_1, \ldots, x_p) \equiv 0$$

where $\frac{\partial}{\partial x_j}$ means differentiation with respect to both appearances of x_j and $m_{p-2}(\hat{x_k}, \hat{x_l})$ means that the variables x_k and x_l are excluded; m_{p-2} is a function of the other $p-2$ space variables.

$\tilde{u}(t,x) = -u(t,-x)$ and $\tilde{\zeta}(t,x) = \zeta(t,-x)$. Then, it is straightforward to compute that

$$\begin{cases} \partial_t \tilde{u} = (\frac{\epsilon}{2} \tilde{u}_{xx} - \frac{1}{2}(\tilde{u}^2)_x) dt + \partial_t \tilde{\zeta}_x \\ \tilde{u}(0,x) \equiv 0. \end{cases}$$

From this, and noting that ζ and $\tilde{\zeta}$ are identically distributed, it follows that

$$\begin{aligned} m_p(t; x_1, \ldots, x_p) &= E_{\mathbf{Q}}\{u(t,x_1) \ldots u(t,x_p)\} \\ &= E_{\mathbf{Q}}\{\tilde{u}(t,x_1) \ldots \tilde{u}(t,x_p)\} = (-1)^p m_p(t; -x_1, \ldots, -x_p). \end{aligned}$$

It follows that $m_{2p}^{(\epsilon)}(t,.)$ is an even function and that $m_{2p+1}^{(\epsilon)}(t,.)$ is an odd function for each integer $p \geq 0$. That is, for each integer $p \geq 1$,

$$m_{2p}^{(\epsilon)}(t; x_1, \ldots, x_{2p}) = m_{2p}^{(\epsilon)}(t; -x_1, \ldots, -x_{2p}) \qquad \forall \epsilon \geq 0, \quad \mathbf{x} \in \mathbf{R}^{2p}, \quad t \in \mathbf{R}_+ \tag{58}$$

and

$$m_{2p+1}^{(\epsilon)}(t; x_1, \ldots, x_{2p+1}) = -m_{2p+1}(t; -x_1, \ldots, -x_{2p+1}) \qquad \forall \epsilon \geq 0, \quad \mathbf{x} \in \mathbf{R}^{2p+1}, \quad t \in \mathbf{R}_+. \tag{59}$$

In particular, $m_{2p+1}^{(\epsilon)}(t, \mathbf{0}) \equiv 0$ for all $t \geq 0$ and all $p \geq 0$. This, together with the upper and lower bounds on $m_p^{(\epsilon)}(t, \mathbf{0})$ uniform in ϵ and together with lemma 15 gives that for each integer $p \geq 0$ and all $T \geq 0$,

$$\limsup_{0 \leq \epsilon_1 \leq \epsilon_2 \to 0} \sup_{0 \leq t \leq T} |m_p^{(\epsilon_1)}(t, \mathbf{0}) - m_p^{(\epsilon_2)}(t, \mathbf{0})| = 0.$$

Set $M_p^{(\epsilon)}(t) := m_p^{(\epsilon)}(t; 0, \ldots, 0)$.

Lemma 16. *For all non negative integer p and for all $t \in \mathbf{R}_+$, the limit $M_p(t) := \lim_{\epsilon \to 0} M_p^{(\epsilon)}(t)$ is well defined.*

Proof of Lemma 16 Firstly, by theorem 11, $\sup_{0<\epsilon<1} \sup_{0 \le t \le T} |M_p^{(\epsilon)}(t)| < K(p,T)$, where $K(p,T)$ is defined on the right hand side of inequality (36). Consider p odd; that is $p = 2q+1$ for non negative integer q. Then $M_{2q+1}^{(\epsilon)}(t) \equiv 0$, so that $M_{2q+1}(t) := \lim_{\epsilon \to 0} M_{2q+1}^{(\epsilon)}(t) = 0$. Secondly, consider p even; that is $p = 2q$ for non negative integer q. Then $M_{2q}^{(\epsilon)}(t) \ge 0$ for all $\epsilon > 0$ and all $t \in \mathbf{R}$. Let

$$\overline{M}_{2q}^{(\epsilon)} = \sup_{0<\delta\le\epsilon} M_{2q}^{(\delta)}(t), \qquad \underline{M}_{2q}^{(\epsilon)} = \inf_{0<\delta\le\epsilon} M_{2q}^{(\delta)}(t).$$

Then, from lemma 15, which states that for each $t > 0$, there is a constant $C(q,T) < +\infty$ such that $\sup_{\epsilon>0} \frac{\partial}{\partial\epsilon} M_{2q}^{(\epsilon)}(t) < C(q,T)$ for all $t \in (0,T)$, it follows that

$$\underline{M}_{2q}^{(\epsilon)} \le \overline{M}_{2q}^{(\epsilon)} \le \underline{M}_{2q}^{(\epsilon)} + \epsilon C(q,T),$$

from which

$$\lim_{\epsilon\to 0} \underline{M}_{2q}^{(\epsilon)}(t) = \lim_{\epsilon\to 0} \overline{M}_{2q}^{(\epsilon)}(t) = \lim_{\epsilon\to 0} M_{2q}^{(\epsilon)}(t).$$

□

Set

$$\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) := m_p^{(\epsilon)}(t; \epsilon x_1, \ldots, \epsilon x_p). \tag{60}$$

Let $K(p,T)$ denote the uniform Lipschitz constant for $(m^{(\epsilon)}(t,.))_{0\le\epsilon\le 1, 0\le t\le T}$ found in theorem 13. That is, for the remainder of the article, $K(p,T)$ will denote a finite positive constant such that

$$\begin{aligned}
&\sup_{0\le t\le T} \sup_{j\in\{1,\ldots,p\}} \sup_{(x_1,\ldots,x_p)\in\mathbf{R}^p} \left| \frac{\partial}{\partial x_j} m_p^{(\epsilon)}(t; x_1, \ldots, x_p) \right| \\
&\quad \le \sup_{0\le t\le T} \sup_{(x_1,\ldots,x_p)\in\mathbf{R}^p} E_{\mathbf{Q}} \left\{ \left| u_x^{(\epsilon)}(t,x_1) u^{(\epsilon)}(t,x_2) \ldots u^{(\epsilon)}(t,x_p) \right| \right\} \le K(p,T)
\end{aligned} \tag{61}$$

Note that, for any fixed $x_1, \ldots, x_p$,

$$\begin{aligned}
&\lim_{\epsilon\to 0} |\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) - M_p(t)| \\
&\quad \le \lim_{\epsilon\to 0} |\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) - M_p^{(\epsilon)}(t)| + \lim_{\epsilon\to 0} |M_p^{(\epsilon)}(t) - M_p(t)| \\
&\quad \le \lim_{\epsilon\to 0} \epsilon (\sum_{j=1}^p |x_j|) K(p,T) + 0 = 0.
\end{aligned}$$

Set

$$\phi_p^{(\epsilon)}(t; x_1, \ldots, x_p) := \frac{\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) - M_p^{(\epsilon)}(t)}{\epsilon}. \tag{62}$$

Note that, by equations (59) and (58), $\phi^{(\epsilon)}_{2p+1}$ is an odd function and $\phi^{(\epsilon)}_{2p}$ is an even function for all integer $p \geq 1$, all $\epsilon \geq 0$ all $t \in \mathbf{R}_+$. That is,

$$\phi^{(\epsilon)}_{2p+1}(t; x_1, \ldots, x_{2p+1}) = -\phi_{2p+1}(t; -x_1, \ldots, -x_{2p+1}) \qquad \forall \epsilon \geq 0, \quad \mathbf{x} \in \mathbf{R}^{2p+1}, \quad t \in \mathbf{R}_+ \tag{63}$$

and

$$\phi^{(\epsilon)}_{2p}(t; x_1, \ldots, x_{2p}) = \phi^{(\epsilon)}_{2p}(t; -x_1, \ldots, -x_{2p}) \qquad \forall \epsilon \geq 0, \quad \mathbf{x} \in \mathbf{R}^{2p}, \quad t \in \mathbf{R}_+. \tag{64}$$

Lemma 17. *It holds that*

$$\sup_{0 \leq t \leq T} \sup_{0 < \epsilon \leq 1} \sup_{x_1, \ldots, x_p} \frac{|\phi^{(\epsilon)}_p(t; x_1, \ldots, x_p)|}{(\sum_{j=1}^p |x_j|)} \leq K(p, T) \tag{65}$$

and, for all $j \in \{1, \ldots, p\}$,

$$\sup_{0 \leq t \leq T} \sup_{0 < \epsilon \leq 1} \sup_{x_1, \ldots, x_p} \limsup_{h \to 0} \frac{|\phi^{(\epsilon)}_p(t; x_1, \ldots, x_j + h, \ldots, x_p) - \phi^{(\epsilon)}_p(t; x_1, \ldots, x_j, \ldots, x_p)|}{|h|} \leq K(p, T) \tag{66}$$

where the existence of a constant $K(p, T)$ independent of ϵ, is guaranteed by theorem 13.

Proof This is an immediate consequence of Taylor's expansion theorem, together with theorem 13. In the second part, for example,

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left| \frac{\phi^{(\epsilon)}_p(t; x_1, \ldots, x_j + h, \ldots, x_p) - \phi^{(\epsilon)}_p(t; x_1, \ldots, x_p)}{h} \right| \\ &\quad = \sup_{0 \leq t \leq T} \left| \frac{m^{(\epsilon)}_p(t; \epsilon x_1, \ldots, \epsilon x_j + \epsilon h, \ldots, \epsilon x_p) - m^{(\epsilon)}_p(t; \epsilon x_1, \ldots, \epsilon x_p)}{\epsilon h} \right| \\ &\quad \leq K(p, T). \end{aligned}$$

Equation (65) also follows directly from the definition, using

$$|\phi^{(\epsilon)}_p(t; x_1, \ldots, x_p)| = |\frac{1}{\epsilon}(m^{(\epsilon)}(t; \epsilon x_1, \ldots, \epsilon x_p) - m^{(\epsilon)}(t; 0, \ldots, 0))| \leq K(p, T) \sum_{j=1}^p |x_j|$$

by Taylor's expansion theorem. $\square$.

Let $\Phi^{(\epsilon)}_p : \mathbf{R}_+ \times \mathbf{R}^p \to \mathbf{R}$ be used to denote the function

$$\Phi^{(\epsilon)}_p(t; .) = \int_0^t \phi^{(\epsilon)}_p(\alpha, .) d\alpha. \tag{67}$$

Lemma 18. *M_p is Lipschitz. That is, for each $T < +\infty$, there exists a constant $C(p,T)$ such that*

$$\sup_{0 \le t \le T} \left(\limsup_{h \to 0} \frac{|M_p(t+h) - M_p(t)|}{h} \right) \le C(p,T).$$

Proof of lemma 18 Note that

$$\frac{\partial}{\partial t}\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) = \frac{1}{2}\Delta_{\mathbf{x}}\phi_p^{(\epsilon)}(t; x_1, \ldots, x_p)$$
$$-\frac{1}{2}\sum_{j=1}^{p} \frac{\partial}{\partial x_j}\phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) + \sum_{j<k}(-\Gamma''(\epsilon(x_j - x_k))\mu^{(\epsilon)}(t; \hat{x_j}, \hat{x_k}).$$

This may be rearranged as

$$\frac{1}{2}\Delta\phi_p^{(\epsilon)}(t; x_1, \ldots, x_p) = \frac{1}{2}\sum_{j=1}^{p} \frac{\partial}{\partial x_j}\phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j)$$
$$+ \left\{ \frac{\partial}{\partial t}\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p) - \sum_{j<k}(-\Gamma''(\epsilon(x_j - x_k))\mu_{p-2}^{(\epsilon)}(t; \hat{x_j}, \hat{x_k}) \right\}. \tag{68}$$

With a change of notation from earlier, set

$$P_s f(\mathbf{x}) = \int_{\mathbf{R}^p} \frac{1}{(2\pi s)^{p/2}} \exp\left\{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2s}\right\} f(\mathbf{y})d\mathbf{y}$$

and set $p(r; \mathbf{z}) = \frac{1}{(2\pi r)^{p/2}} \exp\left\{-\frac{|\mathbf{z}|^2}{2r}\right\}$. By integrating all terms of equation (68) against the test function $\frac{1}{s}\int_0^s p(r; \mathbf{x}-\mathbf{y})dr$, it follows that, for all $s > 0$,

$$\frac{1}{s}\int_{\mathbf{R}^p}\int_0^s p(r; \mathbf{x}-\mathbf{y})\frac{1}{2}\Delta\phi_p^{(\epsilon)}(t; \mathbf{y})drd\mathbf{y} - \frac{1}{2s}\sum_{j=1}^{p}\int_{\mathbf{R}^p}\int_0^s \frac{\partial}{\partial x_j}p(r; \mathbf{x}-\mathbf{y})\phi_{p+1}^{(\epsilon)}(t; y_1, \ldots, y_p, y_j)drd\mathbf{y}$$
$$= \frac{\partial}{\partial t}\left(\frac{1}{s}\int_0^s P_r\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p)dr\right) - \sum_{j<k}\frac{1}{s}\int_0^s P_r(-\Gamma''(\epsilon(x_j - x_k))\mu_{p-2}^{(\epsilon)}(t; \hat{x}_j, \hat{x}_k))dr.$$

Note that $\lim_{\epsilon \to 0}\frac{1}{s}\int_0^s P_r\mu_p^{(\epsilon)}(t; x_1, \ldots, x_p)dr = M_p(t)$ and that

$$\lim_{\epsilon \to 0}\sum_{j<k}\frac{1}{s}\int_0^s P_r(-\Gamma''(\epsilon(x_j - x_k))\mu_{p-2}^{(\epsilon)}(t; \hat{x}_j, \hat{x}_k))dr = \frac{p(p-1)}{2}(-\Gamma''(0))M_{p-2}(t).$$

Since $\frac{1}{2}\Delta$ is the infinitesimal generator of P_s, it follows that for all $s > 0$,

$$\frac{1}{s}\int_{\mathbf{R}^p}\int_0^s p(r; \mathbf{x}-\mathbf{y})\frac{1}{2}\Delta\phi_p^{(\epsilon)}(t; \mathbf{y})drd\mathbf{y} = \frac{P_s\phi_p^{(\epsilon)}(t, \mathbf{x}) - \phi_p^{(\epsilon)}(t, \mathbf{x})}{s}. \tag{69}$$

The observation that equation (69) holds for all $s > 0$ has to be made. When the article did not have this, it received a referee report stating that equation (69) did not hold for all $s > 0$; the referee stated that equation (69) only held in the limit as $s \to 0$. The referee therefore assumed that $\lim_{s\to 0}$ was intended and that the author had made a 'flagrant error'. This was from a respectable journal and the editor sent the author an electronic mail assuring him that the referee was 'an expert in the field' ('the field' was left undefined).

Since $p(s;\mathbf{x}) = \frac{1}{(2\pi s)^{p/2}} \exp\left\{-\frac{|\mathbf{x}|^2}{2s}\right\}$, it follows that for any $s > 0$ and any bounded continuous function $f : \mathbf{R}^p \to \mathbf{R}$,

$$\frac{\partial}{\partial x_j}\int_{\mathbf{R}^p}\frac{1}{(2\pi s)^{p/2}}e^{-|\mathbf{x}-\mathbf{y}|^2/2s}f(y_1,\ldots,y_p)d\mathbf{y} = -\int_{\mathbf{R}^p}\frac{x_j - y_j}{s}p(s,\mathbf{x}-\mathbf{y})f(y_1,\ldots,y_p)d\mathbf{y}.$$

It follows that for all $h > 0$ and $0 < t < T - h$,

$$\begin{aligned}
M_p(t+h) - M_p(t) &= \frac{p(p-1)}{2}(-\Gamma''(0))\int_t^{t+h} M_{p-2}(\tau)d\tau \\
&+ \lim_{\epsilon\to 0}\left(\frac{1}{s}\int_t^{t+h}(P_s\phi_p^{(\epsilon)}(\tau,\mathbf{x}) - \phi_p^{(\epsilon)}(\tau,\mathbf{x}))d\tau\right. \\
&\left.-\frac{1}{2s}\sum_{j=1}^p\int_{\mathbf{R}^p}\int_0^s\frac{(x_j-y_j)}{r}p_r(\mathbf{x}-\mathbf{y})\int_t^{t+h}\phi_{p+1}^{(\epsilon)}(\tau;y_1,\ldots,y_p,y_j)d\tau d\mathbf{y}\right).
\end{aligned} \tag{70}$$

The estimate from lemma 17 (namely, that $\sup_{0\le t\le T}|\phi_p^{(\epsilon)}(t,\mathbf{x})| \le K(p,T)\sum_j |x_j|$) gives

$$\begin{aligned}
\sup_{0<\epsilon\le 1}|P_s\phi_p^{(\epsilon)}(t,\mathbf{x})| &\le K(p,T)\sum_j\int_{\mathbf{R}^p}|y_j|p(s;\mathbf{x}-\mathbf{y})d\mathbf{y} \\
&\le K(p,T)\sum_j\int_{\mathbf{R}^p}(|x_j-y_j| + |x_j|)p(s;\mathbf{x}-\mathbf{y})d\mathbf{y} \\
&= K(p,T)\left(\sum_j |x_j|\right) + \sqrt{\frac{2s}{\pi}}pK(p,T).
\end{aligned} \tag{71}$$

Similarly,

$$
\begin{aligned}
&\sup_{0<\epsilon\leq 1}\left|\sum_{j=1}^{p}\int_{\mathbf{R}^p}\int_0^s \frac{(x_j-y_j)}{r}p_r(\mathbf{x}-\mathbf{y})\phi^{(\epsilon)}_{p+1}(y_1,\ldots,y_p,y_j)d\mathbf{y}\right| \\
&\leq K(p+1,T)\sum_{j=1}^{p}\int_{\mathbf{R}^q}\int_0^s \frac{|x_j-y_j|}{r}p(r;\mathbf{x}-\mathbf{y})\{2|y_j|+\sum_{k\neq j}|y_k|\}d\mathbf{y} \\
&\leq K(p+1,T)\sum_{j=1}^{p}\left\{2|x_j|+\sum_{j\neq k}|x_k|\right\}\int_{\mathbf{R}^p}\int_0^s \frac{|x_j-y_j|}{r}p(r;\mathbf{x}-\mathbf{y})d\mathbf{y} \\
&\quad +K(p+1,T)\sum_{j=1}^{p}\int_{\mathbf{R}^p}\int_0^s \frac{|y_j|}{r}p(r;\mathbf{y})\left\{2|y_j|+\sum_{k\neq j}|y_k|\right\}d\mathbf{y} \\
&= \frac{2\sqrt{2}(p+1)}{\sqrt{\pi}}K(p+1,T)\left(\sum_j |x_j|\right)\sqrt{s}+2pK(p+1,T)\left(1+\frac{p-1}{\pi}\right)s.
\end{aligned}
$$

Now, using the upper bound, uniform in ϵ given by equation (36) in theorem 11, it follows that for $t\in[0,T]$, there is a constant $c(p,T)<+\infty$ such that $\sup_{0\leq t\leq T}|M_{2(p-1)}(t)|\leq c(p,T)$.
Taking $s\to+\infty$, it follows from equation (70) that

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\limsup_{h\to 0}\left|\frac{M_{2p}(t+h)-M_{2p}(t)}{h}\right| \\
&\qquad \leq p(2p-1)(-\Gamma''(0))c(p,T)+4pK(2p+1,T)\left(1+\frac{4(2p-1)}{\pi}\right).
\end{aligned} \tag{72}
$$

Lemma 18 follows. ☐

Lemma 19. *Let $\Phi^{(\epsilon)}_{p+1}$ be defined according to equation (67). Then for all $p\geq 2$*

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\left|M_p(t)-\frac{p(p-1)}{2}\int_0^t M_{p-2}(\alpha)d\alpha\right. \\
&\quad\left.-\lim_{s\to+\infty}\frac{1}{2s}\sum_{j=1}^{p}\int_0^s\int_{\mathbf{R}^p}\frac{x_j-y_j}{r}p(r,\mathbf{x}-\mathbf{y})\Phi^{(\epsilon)}_{p+1}(t;y_1,\ldots,y_p,y_j)d\mathbf{y}dr\right|=0.
\end{aligned}
$$

Proof From equation (70), it follows that

$$\lim_{\epsilon\to 0}\Bigg|M_p(t)-\frac{p(p-1)}{2}\int_0^t M_{p-2}(\alpha)d\alpha$$
$$-\frac{\int_0^t(P_s\phi_p^{(\epsilon)}(\alpha,\mathbf{x})-\phi_p^{(\epsilon)}(\alpha,\mathbf{x}))d\alpha}{s} \tag{73}$$
$$-\frac{1}{2s}\sum_{j=1}^p\int_0^s\int_{\mathbf{R}^p}\frac{x_j-y_j}{r}p(r,\mathbf{x}-\mathbf{y})\left\{\int_0^t\phi_{p+1}^{(\epsilon)}(\alpha;y_1,\ldots,y_p,y_j)d\alpha\right\}d\mathbf{y}dr\Bigg|=0.$$

This holds for all $s>0$. Recall equation (67). Using equations (65) and (71), it follows that for all $0<t<T<+\infty$

$$\left|\frac{\int_0^t(P_s\phi_p^{(\epsilon)}(\alpha,\mathbf{x})-\phi_p^{(\epsilon)}(\alpha,\mathbf{x}))d\alpha}{s}\right|\le\frac{2}{s}TK(p,T)\sum_{j=1}^p|x_j|+\frac{1}{\sqrt{s}}\sqrt{\frac{2}{\pi}}pTK(p,T),$$

so that

$$\lim_{\epsilon\to 0}\Bigg|M_p(t)-\frac{p(p-1)}{2}\int_0^t M_{p-2}(\alpha)d\alpha$$
$$-\frac{1}{2s}\sum_{j=1}^p\int_0^s\int_{\mathbf{R}^p}\frac{x_j-y_j}{r}p(r,\mathbf{x}-\mathbf{y})\Phi_{p+1}^{(\epsilon)}(t;y_1,\ldots,y_p,y_j)d\mathbf{y}dr\Bigg|$$
$$\le\frac{2}{s}TK(p,T)\sum_{j=1}^p|x_j|+\frac{1}{\sqrt{s}}\sqrt{\frac{2}{\pi}}pTK(p,T). \tag{74}$$

Now, letting $s\to+\infty$, the result follows. □

Proposition 20. *Let $p\in\mathbf{Z}_+$ (the non negative integers) let $\Phi_{p+1}^{(\epsilon)}$, be the function defined in equation (67). Then, for all $T\in(0,\infty)$ and any bounded domain $D\in\mathbf{R}^p$,*

$$\sup_{0<\epsilon<1}\lim_{s\to+\infty}\sup_{0\le t\le T}\sup_{\mathbf{x}\in D}\left|\sum_{j=1}^p\frac{\partial}{\partial x_j}\int_{\mathbf{R}^p}\frac{1}{(2\pi s)^{p/2}}e^{-|\mathbf{x}-\mathbf{y}|^2/2s}\Phi_{p+1}^{(\epsilon)}(t;y_1,\ldots,y_p,y_j)d\mathbf{y}\right|=0.$$

In the following, the t in the notation will be suppressed; $\Phi_{p+1}^{(\epsilon)}(x_1,\ldots,x_{p+1})$ will be used to denote $\Phi_{p+1}^{(\epsilon)}(t;x_1,\ldots,x_{p+1})$ and it will be assumed that $0<t<T$ where $T<+\infty$.

Proof of Proposition 20 Set

$$\tilde{\Phi}_{p+1}^{(\epsilon)}(s;x_1,\ldots,x_{p+1})=\int_{\mathbf{R}^{p-1}}\frac{1}{(2\pi s)^{(p-1)/2}}e^{-\frac{1}{2s}\sum_{j=1}^{p-1}y_j^2}\Phi_{p+1}^{(\epsilon)}(x_1+y_1,\ldots,x_{p-1}+y_{p-1},x_p,x_{p+1})d\mathbf{y}$$

and set

$$
\begin{aligned}
\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_{p+1}) &= \int_{-\infty}^{\infty}\frac{1}{(2\pi s)^{1/2}}e^{-z^2/2s}\tilde{\Phi}^{(\epsilon)}_{p+1}(s;x_1+z,\ldots,x_{p-1}+z,x_p,x_{p+1})dz \\
&= \int_{-\infty}^{\infty}\frac{1}{(2\pi s)^{1/2}}e^{-z^2/2s}\tilde{\Phi}^{(\epsilon)}_{p+1}(s;x_1,\ldots,x_{p-1},x_p+z,x_{p+1}+z)dz.
\end{aligned}
$$

Using the bound in equation (66), together with equation (62), it is straightforward that

$$
\sup_{0\le\epsilon<1}\sup_{\mathbf{x}\in\mathbf{R}^{p+1}}\left|\frac{\partial}{\partial x_j}\Phi^{(\epsilon)}_{p+1}(x_1,\ldots,x_{p+1})\right| < TK(p+1,T).
$$

From this, it follows directly that for all $(j,k)\in\{1,\ldots,p-1\}^2$,

$$
\begin{aligned}
&\sup_{0<\epsilon<1}\sup_{\mathbf{x}\in\mathbf{R}^{p+1}}\left|\frac{\partial^2}{\partial x_j\partial x_k}\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_{p+1})\right| \\
&\quad= \sup_{0<\epsilon<1}\sup_{\mathbf{x}\in\mathbf{R}^{p+1}}\left|\frac{\partial^2}{\partial x_j\partial x_k}\int_{\mathbf{R}}\frac{1}{(2\pi s)^{1/2}}e^{-z^2/2s}\int_{\mathbf{R}^{p-1}}\frac{1}{(2\pi s)^{(p-1)/2}}e^{-\frac{1}{2s}\sum_{j=1}^{p-1}(x_j-y_j-z)^2}\right. \\
&\qquad\qquad\left.\times\,\Phi^{(\epsilon)}_{p+1}(y_1,\ldots,y_{p-1},x_p,x_{p+1})d\mathbf{y}dz\right| \\
&\quad\le \sup_{0<\epsilon<1}\sup_{\mathbf{x}\in\mathbf{R}^{p+1}}\int_{\mathbf{R}}\frac{1}{(2\pi s)^{1/2}}e^{-z^2/2s}\int_{\mathbf{R}^{p-1}}\frac{1}{(2\pi s)^{(p-1)/2}}e^{-\frac{1}{2s}\sum_{j=1}^{p-1}(x_j-y_j-z)^2} \\
&\qquad\qquad\times\left|\frac{x_k-y_k-z}{s}\right|\left|\frac{\partial}{\partial y_j}\Phi^{(\epsilon)}_{p+1}(y_1,\ldots,y_{p-1},x_p,x_{p+1})\right|d\mathbf{y}dz \\
&\quad\le TK(p+1,T)\int_{-\infty}^{\infty}\frac{|y|}{s}\frac{1}{(2\pi s)^{1/2}}e^{-|y|^2/2s}dy \\
&\quad= \sqrt{\frac{2}{\pi s}}TK(p+1,T)\stackrel{s\to\infty}{\longrightarrow}0. \qquad (75)
\end{aligned}
$$

Set

$$
\gamma^{(\epsilon)}(s;x_2,\ldots,x_{p-1},x_p,x_{p+1}) = \left.\frac{\partial}{\partial x_1}\psi^{(\epsilon)}(s;x_1,x_2,\ldots,x_{p-1},x_p,x_{p+1})\right|_{x_1=0},
$$

then it follows directly from (75) that

$$
\lim_{s\to+\infty}\sup_{(x_2,\ldots,x_{p+1})\in\mathbf{R}^p}\max_{j\in\{2,\ldots,p-1\}}\sup_{0<\epsilon<1}\left|\frac{\partial}{\partial x_j}\gamma^{(\epsilon)}(s;x_2,\ldots,x_{p-1},x_p,x_{p+1})\right| = 0 \qquad (76)
$$

and hence that for any bounded $D\subset\mathbf{R}^{p-1}$ and all $(x_p,x_{p+1})\in\mathbf{R}^2$,

$$
\lim_{s\to+\infty}\sup_{(x_2,\ldots,x_{p-1})\in D}\sup_{0<\epsilon<1}\left|\gamma^{(\epsilon)}(s;x_2,\ldots,x_{p-1},x_p,x_{p+1})-\gamma^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1})\right| = 0. \qquad (77)
$$

From equation (76), it follows by Taylor's expansion theorem that for any bounded subset $D\subset\mathbf{R}^{p-1}$,

$$
\begin{aligned}
\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_{p+1}) &= \psi^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1}) \\
&= \gamma^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1})\sum_{j=1}^{p-1} x_j + \frac{1}{2}\sum_{j,k=1}^{p-1} \partial^2_{jk}\psi^{(\epsilon)}(s;x_1^*,\ldots,x_{p-1}^*,x_p,x_{p+1})
\end{aligned}
$$

where $|x_j^*| \leq |x_j|$ for $j = 1,\ldots,p-1$. It follows from (75) that for any bounded $D \subset \mathbf{R}^{p+1}$,

$$
\lim_{s\to+\infty}\sup_{\mathbf{x}\in D}\sup_{0<\epsilon<1}\left|\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_{p+1}) - \gamma^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1})\sum_{j=1}^{p-1} x_j\right| = 0. \tag{78}
$$

Taylor's expansion theorem applied to the first $p-1$ variables of $\partial\psi^{(\epsilon)}(x_1,\ldots,x_{p-1},x_p,x_{p+1})$ gives that there are points $x_1^*,\ldots,x_{p-1}^*$ such that $0 \leq |x_j^*| \leq |x_j|$ such that

$$
\partial_j\psi^{(\epsilon)}(x_1,\ldots,x_{p-1},x_p,x_{p+1}) = \gamma^{(\epsilon)}(0,\ldots,0,x_p,x_{p+1}) + \sum_{k=1}^{p-1} x_k\partial^2_{jk}\psi^{(\epsilon)}(x_1^*,\ldots,x_{p-1}^*,x_p,x_{p+1})
$$

from which it follows, using equations (75) and (77), that for any bounded set $D \subset \mathbf{R}^{p+1}$,

$$
\lim_{s\to+\infty}\sup_{\mathbf{x}\in D}\sup_{0<\epsilon<1}\left|\frac{\partial}{\partial x_j}\psi^{(\epsilon)}(x_1,\ldots,x_{p-1},x_p,x_{p+1}) - \gamma^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1})\right| = 0, \qquad j \in \{2,\ldots,p-1\}.
$$

By construction, $\psi(s;x_1,\ldots,x_{p-1},x_p,x_{p+1}) = \psi(s;x_{\sigma(1)}+a,\ldots,x_{\sigma(p-1)}+a,x_{p+1}+a,x_p+a)$ for any permutation σ of $\{1,\ldots,p-1\}$ and any $a \in \mathbf{R}$ and all $(x_1,\ldots,x_{p+1}) \in \mathbf{R}^{p+1}$. From equation (78), it follows that for any bounded $D \subset \mathbf{R}^{p+2}$,

$$
\lim_{s\to+\infty}\sup_{0<\epsilon<1}\sup_{(x_1,\ldots,x_{p+1},a)\in D} \left|\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_{p+1}) - \gamma^{(\epsilon)}(s;0,\ldots,0,a+x_p,a+x_{p+1})\sum_{j=1}^{p-1}(a+x_j)\right| = 0. \tag{79}
$$

It follows that

$$
\begin{aligned}
\lim_{s\to+\infty}\sup_{0<\epsilon<1}\sup_{(x_1,\ldots,x_{p+1},a)\in D}\Bigg|\Big(\gamma^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1}) - \gamma^{(\epsilon)}(s;0,\ldots,0,x_p+a,x_{p+1}+a)\Big)\sum_{j=1}^{p-1} x_j \\
-(p-1)a\gamma^{(\epsilon)}(s;0,\ldots,0,x_p+a,x_{p+1}+a)\Bigg| = 0.
\end{aligned}
$$

From this, it follows (by considering the coefficient of $\sum_{j=1}^{p-1} x_j$) that

$$\lim_{s\to+\infty}\sup_{0<\epsilon<1}\left|\gamma^{(\epsilon)}(s;0,\ldots,x_p,x_{p+1})-\gamma^{(\epsilon)}(s;0,\ldots,0,x_p+a,x_{p+1}+a)\right|=0 \qquad \forall(a,x_p,x_{p+1})\in\mathbf{R}^3$$

and hence that

$$\lim_{s\to+\infty}\sup_{0<\epsilon<1}\left|\gamma^{(\epsilon)}(s;0,\ldots,0,0,z)-\gamma^{(\epsilon)}(s;0,\ldots,0,x,x+z)\right|=0 \qquad \forall(x,z)\in\mathbf{R}^2.$$

Using this, it follows by considering the term $(p-1)a\gamma^{(\epsilon)}(s;0,\ldots,0,x_p+a,x_{p+1}+a)$, that

$$\lim_{s\to+\infty}\sup_{0<\epsilon<1}|\gamma^{(\epsilon)}(s;0,\ldots,0,x_p,x_{p+1})|=0 \qquad \forall(x_p,x_{p+1})\in\mathbf{R}^2,$$

from which it follows that for any bounded domain $D\subset\mathbf{R}^{p+1}$ and all $j\in\{1,\ldots,p-1\}$

$$\lim_{s\to+\infty}\sup_{\mathbf{x}\in D}\sup_{0<\epsilon<1}\left|\frac{\partial}{\partial x_j}\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_{p+1})\right|=0.$$

Now note that

$$\frac{\partial}{\partial x_p}\int_{\mathbf{R}^p}\frac{1}{(2\pi s)^{p/2}}e^{-|\mathbf{x}-\mathbf{y}|^2/2s}\Phi_{p+1}^{(\epsilon)}(y_1,\ldots,y_p,y_p)d\mathbf{y}=\frac{d}{dx_p}\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_p)$$

where the differential $\frac{d}{dx_p}$ refers to both appearances of the variable x_p and, since

$$\psi^{(\epsilon)}(s;x_1,\ldots,x_{p-1},x_p,x_p)=\psi^{(\epsilon)}(s;x_1-x_p,\ldots,x_{p-1}-x_p,0,0),$$

it follows that

$$\frac{\partial}{\partial x_p}\int_{\mathbf{R}^p}\frac{1}{(2\pi s)^{p/2}}e^{-|\mathbf{x}-\mathbf{y}|^2/2s}\Phi_{p+1}^{(\epsilon)}(y_1,\ldots,y_p,y_p)d\mathbf{y}=-\sum_{j=1}^{p-1}\frac{\partial}{\partial x_j}\psi^{(\epsilon)}(s;x_1-x_p,\ldots,x_{p-1}-x_p,0,0),$$

from which, for any bounded $D\subset\mathbf{R}^p$,

$$\lim_{s\to+\infty}\sup_{0<\epsilon<1}\sup_{\mathbf{x}\in D}\left|\frac{\partial}{\partial x_p}\int_{\mathbf{R}^p}\frac{1}{(2\pi s)^{p/2}}e^{-|\mathbf{x}-\mathbf{y}|^2/2s}\Phi_{p+1}^{(\epsilon)}(y_1,\ldots,y_p,y_p)d\mathbf{y}\right|=0.$$

Let

$$\tau_j(k)=\begin{cases} k & k\neq j\\ p & k=j\\ j & k=p\end{cases}$$

The result now follows by noting that for $j \in \{1, \ldots, p\}$,

$$\Phi_{p+1}^{(\epsilon)}(x_1, \ldots, x_p, x_j) = \Phi_{p+1}^{(\epsilon)}(x_{\tau_j(1)}, \ldots, x_{\tau_j(p)}, x_j),$$

from which it follows that

$$\sup_{0<\epsilon<1} \lim_{s \to +\infty} \sup_{0 \le t \le T} \sup_{\mathbf{x} \in D} \left| \frac{\partial}{\partial x_j} \int_{\mathbf{R}^p} \frac{1}{(2\pi s)^{p/2}} e^{-|\mathbf{x}-\mathbf{y}|^2/2s} \Phi_{p+1}^{(\epsilon)}(t; y_1, \ldots, y_p, y_j) d\mathbf{y} \right| = 0$$

for each $j = 1, \ldots, p$ and hence that

$$\sup_{0<\epsilon<1} \lim_{s \to +\infty} \sup_{0 \le t \le T} \sup_{\mathbf{x} \in D} \left| \sum_{j=1}^{p} \frac{\partial}{\partial x_j} \int_{\mathbf{R}^p} \frac{1}{(2\pi s)^{p/2}} e^{-|\mathbf{x}-\mathbf{y}|^2/2s} \Phi_{p+1}^{(\epsilon)}(t; y_1, \ldots, y_p, y_j) d\mathbf{y} \right| = 0.$$

$\square$

Proof of theorem 1 Lemma 19 gave

$$\begin{aligned} &\left| M_p(t) - \frac{p(p-1)}{2}(-\Gamma''(0)) \int_0^t M_{p-2}(\alpha) d\alpha \right| \\ &\qquad \le \limsup_{\epsilon \to 0} \left| \lim_{s \to +\infty} \frac{1}{2s} \left(\sum_{j=1}^{p} \int_0^s P_r \left(\frac{d}{dx_j} \Phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) \right) dr \right) \right|. \end{aligned}$$

By proposition 20, it follows that for each $r > 0$ and each $T < +\infty$, $\mathbf{x} \in \mathbf{R}^p$,

$$\lim_{s \to +\infty} \sup_{0 \le t \le T} \sup_{0<\epsilon<1} \left| P_{sr} \frac{d}{dx_j} \Phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) \right| = 0.$$

This, together with the the uniform bound

$$\sup_{0 \le t \le T} \sup_{(x_1, \ldots, x_p) \in \mathbf{R}^p} \left| P_{sr} \frac{d}{dx_j} \Phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) \right| \le 2TK(p+1, T)$$

and the bounded convergence theorem, imply that

$$\begin{aligned} &\lim_{s \to +\infty} \frac{1}{2s} \int_0^s P_r \frac{d}{dx_j} \Phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) dr \\ &\quad = \lim_{s \to +\infty} \int_0^1 P_{sr} \frac{d}{dx_j} \Phi_{p+1}^{(\epsilon)}(t; x_1, \ldots, x_p, x_j) dr \\ &\quad = 0. \end{aligned}$$

It follows that

$$M_p(t) = \frac{p(p-1)}{2}(-\Gamma''(0)) \int_0^t M_{p-2}(\alpha) d\alpha. \tag{80}$$

Since $M_1(t) \equiv 0$ and $M_0(t) \equiv 1$, the result follows for $t \in [0, T]$, by solving the system of equations (80). By taking T arbitrarily large, the result holds for all $0 < t < +\infty$. $\square$

Theorem 2 is now considered. Firstly, it is shown that the solutions to the equation (1) converge in L^p norm as $\epsilon \to 0$, for all $p \geq 2$.

Let $\tilde{C} : \mathbf{R}_+ \times \Omega \to \mathbf{R}_+$ denote $\tilde{C}(1, .)$ from equation (43) and recall that for each $t \in \mathbf{R}_+$ solutions to equation (1) satisfy

$$\sup_{0 \leq s \leq t} \sup_{x \in [0, 2\pi)} \sup_{0 < \epsilon < 1} |u^{(\epsilon)}(s, x)| \leq \tilde{C}(t)$$

and that, from the inequality (46), the definition in equation (45) and the computation below equation (46) that $E_{\mathbf{Q}} \left\{ \tilde{C}^p(t) \right\} < +\infty$. The bounds are given in theorem 11. Furthermore, the result of theorem 1 is that

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} E_{\mathbf{Q}} \left\{ \int_0^{2\pi} |u^{(\epsilon)}(t, x)|^{2p} dx \right\} = \left(\prod_{j=1}^{p} (2j - 1) \right) \left(-\Gamma''(0) \right)^p t^p.$$

The following argument shows convergence of $u^{(\epsilon)}$ in norm as $\epsilon \to 0$ in the L^p spaces. The following result is required.

Theorem 21. *Let $v : [0, 1] \times [0, 2\pi] \to \mathbf{R}$ be a process for which there are three strictly positive constants γ, c, δ such that*

$$E_{\mathbf{Q}} \left\{ \left| v^{(\epsilon_1)}(x_1) - v^{(\epsilon_2)}(x_2) \right|^{\gamma} \right\} \leq c(|\epsilon_1 - \epsilon_2| + |x_1 - x_2|)^{2+\delta}$$

then there is a modification $\hat{v}$ of v such that

$$E_{\mathbf{Q}} \left\{ \left(\sup_{(\epsilon_1, x_1) \neq (\epsilon_2, x_2)} \frac{|\hat{v}^{(\epsilon_1)}(x_1) - \hat{v}^{(\epsilon_2)}(x_2)|}{(|x_1 - x_2| + |\epsilon_1 - \epsilon_2|)^{\alpha}} \right)^{\gamma} \right\} < +\infty$$

for all $\alpha \in [0, \frac{\delta}{\gamma})$.

Proof This is a standard result and may be found, for example, as theorem (2.1) on page 26 of Revuz and Yor [4]. There it is presented for $v : [0, 1] \times [0, 1] \to \mathbf{R}$; the rescaling is standard. $\square$

Lemma 22. *Consider the system of equations for $\epsilon > 0$:*

$$\left\{ \begin{array}{l} \partial_t v^{(\epsilon)} = \frac{\epsilon}{2} v^{(\epsilon)}_{xx} dt - \frac{1}{2} u^{(\epsilon)2} dt + \partial_t \zeta \\ v^{(\epsilon)}(0, x) \equiv 0 \end{array} \right. \tag{81}$$

$$\left\{ \begin{array}{l} \partial_t u^{(\epsilon)} = \frac{\epsilon}{2} u^{(\epsilon)}_{xx} dt - \frac{1}{2} (u^{(\epsilon)2})_x dt + \partial_t \zeta_x \\ u^{(\epsilon)}(0, x) \equiv 0 \end{array} \right. \tag{82}$$

The solutions $v^{(\epsilon)}$ and $u^{(\epsilon)}$ to equations (81) and (82) respectively in $\mathcal{S}_p^*$ (defined in equation (12) in the statement of lemma 5) converge in L^p norm to functions v and u respectively, which respectively satisfy

$$\left\{\begin{array}{l} \partial_t v + \frac{1}{2}u^2 dt = \partial_t \zeta \\ v(0,x) \equiv 0 \end{array}\right. \tag{83}$$

and

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}(u^2)_x dt = \partial_t \zeta_x \\ u(0,x) \equiv 0. \end{array}\right. \tag{84}$$

Proof Firstly, theorem 7 gives that for $\epsilon > 0$, equation (82) has a unique solution in $\mathcal{S}_p^*$ for each $p > 0$, hence equation (81) has a unique solution in $\mathcal{S}_p^*$ for each $p > 0$, since once $u^{(\epsilon)}$ is established, equation (81) is linear and existence an uniqueness follows directly in a straightforward manner. Set $\tilde{v}^{(\epsilon)} = \frac{\partial}{\partial \epsilon} v^{(\epsilon)}$. It follows, simply by taking the derivative with respect to ϵ in equation (81) and using $v_x^{(\epsilon)} = u^{(\epsilon)}$ and $v_{xx}^{(\epsilon)} = u_x^{(\epsilon)}$, that

$$\left\{\begin{array}{l} \tilde{v}_t^{(\epsilon)} = \frac{\epsilon}{2}\tilde{v}_{xx}^{(\epsilon)} + \frac{1}{2}u_x^{(\epsilon)} - u^{(\epsilon)}\tilde{v}_x^{(\epsilon)} \\ \tilde{v}^{(\epsilon)}(0,x) \equiv 0. \end{array}\right.$$

Since $\frac{\epsilon}{2}\frac{\partial^2}{\partial x^2} - u^{(\epsilon)}(t,x)\frac{\partial}{\partial x}$ is the infinitesimal generator, given in equation (31) of the process X from definition 8, it follows directly that

$$\tilde{v}^{(\epsilon)}(t,x) = \frac{1}{2}\int_0^t E_{\mathbf{P}}\left[u_x^{(\epsilon)}(s, X_{s,t}(x))\right] ds. \tag{85}$$

Now recall that

$$X_{s,t}^{(\epsilon)}(x) = x + \left(w_t^{(\epsilon)} - w_s^{(\epsilon)}\right) - \int_s^t u^{(\epsilon)}\left(r, X_{r,t}^{(\epsilon)}(x)\right) dr$$

(equation (32)). Taking the derivative in x and suppressing some appearances of ϵ in the notation,

$$X'_{s,t}(x) = 1 - \int_s^t u_x(r, X_{r,t}(x)) X'_{r,t}(x) dr \qquad \forall x \in \mathbf{R},$$

giving

$$X'_{s,t}(x) = \exp\left\{-\int_s^t u_x(r, X_{r,t}(x)) dr\right\}$$

and hence

$$\log X'_{0,t}(x) = -\int_0^t u_x^{(\epsilon)}(r, X_{r,t}(x)) dr.$$

It now follows from equation (85) that

$$\tilde{v}^{(\epsilon)}(t,x) = -\frac{1}{2}E_{\mathbf{P}}[\log X'_{0,t}(x)].$$

It follows that

$$\begin{aligned}
&E_{\mathbf{Q}}\{|\tilde{v}^{(\epsilon)}(t,x)|^{2p}\} \\
&\quad \leq \frac{1}{2^{2p}}\left(E_{\mathbf{Q}}\left\{E_{\mathbf{P}}\left[(\log X'_{0,t}(x))^{2p}\chi_{X'_{0,t}(x)>1}\right]\right\} + E_{\mathbf{Q}}\left\{E_{\mathbf{P}}\left[(\log X'_{0,t}(x))^{2p}\chi_{X'_{0,t}(x)<1}\right]\right\}\right) \\
&\quad = I + II.
\end{aligned}$$

For part I, note that

$$\frac{d}{dx}(\log x)^{2p} = 2p\frac{(\log x)^{2p-1}}{x}$$

and

$$\begin{aligned}
\frac{d^2}{dx^2}(\log x)^{2p} &= \frac{2p(2p-1)(\log x)^{2p-2}}{x^2} - \frac{2p(\log x)^{2p-1}}{x^2}. \\
&= \frac{2p(\log x)^{2(p-1)}}{x^2}\left((2p-1) - \log x\right).
\end{aligned}$$

The maximum of $\frac{d}{dx}(\log x)^{2p}$ in the range $x \in (0,+\infty)$ occurs at e^{2p-1} and is $2p(2p-1)^{2p-1}e^{-(2p-1)}$. It follows that, for $x \in [1,+\infty)$,

$$(\log x)^{2p} \leq 2p(2p-1)^{2p-1}e^{-(2p-1)}(x-1) \leq (2p)^{2p}x.$$

Since $\int_0^{2\pi} X'(x)dx = 2\pi$, $X' \geq 0$ and $E_{\mathbf{Q}}\{|\tilde{v}^{(\epsilon)}(t,x)|^{2p}\} = \frac{1}{2\pi}\int_0^{2\pi} E_{\mathbf{Q}}\{|\tilde{v}^{(\epsilon)}(t,x)|^{2p}\}dx$ and, for $x \geq 1$, $(\log x)^{2p} \leq (2p)^{2p}x$, it follows that

$$I \leq (2p)^{2p}.$$

Set $u_x = \phi$, then ϕ satisfies

$$\begin{cases} \partial_t\phi = \frac{\epsilon}{2}\phi_{xx}dt - \phi^2 dt - u\phi_x dt + \partial_t\zeta_{xx} \\ \phi(0,x) \equiv 0. \end{cases}$$

Note that $u_x = \phi \leq w$, where w satisfies

$$\begin{cases} \partial_t w = \frac{\epsilon}{2}w_{xx}dt - uw_x dt + \partial_t\zeta_{xx} \\ w(0,x) \equiv 0. \end{cases} \tag{86}$$

The solution to equation (86) may be written as

$$\begin{aligned}
w(t,x) &= -\sum_{n\geq 0} n^2 a_n\left(\int_0^t E_{\mathbf{P}}[\cos(nX_{s,t}(x))]d_s\beta^{1n}(s) + \int_0^t E_{\mathbf{P}}[\sin(nX_{s,t}(x))]d_s\beta^{2n}(s)\right) \\
&= \theta(2;t,x)
\end{aligned}$$

where θ is defined in equation (37). Since

$$X'_{0,t}(x) = e^{-\int_0^t u_x(r, X_{t,}(x))dr} \geq e^{-\int_0^t \theta(2; r, X_{t,}(x))dr},$$

it follows that

$$E_{\mathbf{Q}}\{E_{\mathbf{P}}[(\log(X'_{0,t} \wedge 1))^{2p}]\} \leq t^{2p} E_{\mathbf{Q}} \left\{ \left(\sup_{0 \leq s \leq t} \sup_x |\theta(2; s, x)| \right)^{2p} \right\},$$

which is bounded above independently of ϵ by an application of lemma 12, so that

$$II \leq K(2p, t) < +\infty$$

where $K(2p, .)$ is an increasing function such that $K(2p, t) < +\infty$ for each $t < +\infty$, which is independent of ϵ. It follows that for each $T < +\infty$

$$\sup_{0 \leq t \leq T} \sup_{0 < \epsilon \leq 1} E_{\mathbf{Q}} \left\{ \left| \tilde{v}^{(\epsilon)}(t, x) \right|^{2p} \right\} \leq (2p)^{2p} + K(2p, T) < +\infty. \tag{87}$$

Recall the definition of $\|f\|_p(t)$ given in equation (19). From equation (81),

$$\partial_t v^{(\epsilon)p} = \left(\frac{\epsilon p}{2} v^{(\epsilon)p-1} v_{xx}^{(\epsilon)} - \frac{p}{2} v^{(\epsilon)p-1} u^{(\epsilon)2} + \frac{p(p-1)}{2} v^{(\epsilon)p-2} \Gamma(0) \right) dt + p v^{(\epsilon)p-1} \partial_t \zeta.$$

Integration by parts and applications of Hölder's inequality yield that for positive integer p,

$$\begin{aligned} \frac{d}{dt} \|v^{(\epsilon)}\|_{2p}^{2p}(t) &= -\epsilon p(2p-1) E_{\mathbf{Q}} \left\{ \left(\int v^{(\epsilon)2(p-1)} v_x^2 dx \right) \right\} - p E_{\mathbf{Q}} \left\{ \left(\int v^{(\epsilon)2p-1} u^{(\epsilon)2} dx \right) \right\} \\ &\quad + p(2p-1)\Gamma(0) E_{\mathbf{Q}} \left\{ \left(\int v^{(\epsilon)2p-2} dx \right) \right\} \\ &\leq p \|v^{(\epsilon)}\|_{2p}^{2p-1} \|u^{(\epsilon)}\|_{4p}^2 + p(2p-1) \|v^{(\epsilon)}\|_{2p}^{2p-2} \Gamma(0), \end{aligned}$$

so that

$$\begin{cases} \frac{d}{dt} \|v^{(\epsilon)}\|_{2p}^2(t) \leq \|v^{(\epsilon)}\|_{2p} \|u^{(\epsilon)}\|_{4p}^2 + (2p-1)\Gamma(0) \\ \|v^{(\epsilon)}\|_{2p}(0) = 0. \end{cases}$$

It has already been established that, for $T < +\infty$, there is a constant $K(p, T) < +\infty$ such that

$$\sup_{0 \leq t \leq T} \|u\|_{4p}^2(t) \leq E_{\mathbf{Q}}\{\tilde{C}^{4p}(1, T)\}^{1/2p} < K(p, T) < +\infty,$$

where $\tilde{C}(b, t)$ is given by equation (43) and the existence of a finite upper bound $K(p, T)$ follows from lemma 12. It follows that

$$\|v^{(\epsilon)}\|_{2p}^2(t) \leq (1 + (2p-1)\Gamma(0)t) \exp\{K(p, T)t\}$$

and hence that for any $T < +\infty$ and any $p > 1$, $v \in L^p([0,T]\times[0,2\pi]\times\Omega)$. That is, for each there is a positive function $C(p,t)$, increasing in t, with $C(p,t) < +\infty$ if $t < +\infty$ such that

$$|||v^{(\epsilon)}|||_{T,p} := \left(\int_0^T \int_0^{2\pi} E_{\mathbf{Q}}\left\{|v^{(\epsilon)}(t,x)|^p\right\} dxdt\right)^{1/p} < C(p,T).$$

,

Let

$$K_1(p,t) = E_{\mathbf{Q}}\left\{\sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}\sup_{0\leq\epsilon\leq 1}|u^{(\epsilon)}(s,x)|^p\right\}$$

and note that, since $v^{(\epsilon)}(t,x_2) - v^{(\epsilon)}(t,x_1) = \int_{x_1}^{x_2} u^{(\epsilon)}(t,y)dy$, it follows by a standard application of Hölder's inequality that

$$\sup_{0\leq t\leq T}\sup_{0<\epsilon\leq 1} E_{\mathbf{Q}}\left\{|v^{(\epsilon)}(t,x_2) - v^{(\epsilon)}(t,x_1)|^{2p}\right\} \leq |x_2 - x_1|^{2p} K_1(2p,T). \tag{88}$$

Let $K(p,T)$ denote the same quantity as in equation (87). From equation (87), it follows that for all $(\epsilon_1,\epsilon_2) \in (0,1]^2$,

$$\sup_{0\leq t\leq T}\sup_{0\leq x\leq 2\pi} E_{\mathbf{Q}}\left\{|v^{(\epsilon_1)}(t,x) - v^{(\epsilon_2)}(t,x)|^{2p}\right\} \leq |\epsilon_1 - \epsilon_2|^{2p}\left((2p)^{2p} + K(p,t)\right)^{2p}, \tag{89}$$

From equations (88) and (89), it follows by an application of theorem 21 that

$$\lim_{\epsilon\to 0}\sup_{0\leq t\leq T} E_{\mathbf{Q}}\left\{\sup_{0\leq x\leq 2\pi}|v^{(\epsilon)}(t,x) - v(t,x)|^p\right\} = 0.$$

from which

$$|||v^{(\epsilon)} - v|||_{T,p} \stackrel{\epsilon\to 0}{\longrightarrow} 0$$

for each $T < +\infty$ and each $p > 0$. It follows that $\mathbf{Q}$ almost surely, for all $T > 0$, $(m,n) \in \mathbf{Z}^2$, there is a random variable $\lambda_T(m,n)$ such that $E_{\mathbf{Q}}\left\{|\lambda_T(m,n)|^p\right\} < +\infty$ for all $0 < p < +\infty$ and such that

$$\int_0^T \int_0^{2\pi} e^{ism\frac{2\pi}{T} + ixn} v^{(\epsilon)}(s,x)dxdt \to \lambda_T(m,n).$$

Recall that $u^{(\epsilon)} = v_x^{(\epsilon)}$. Also,

$$\sup_{0\leq\epsilon\leq 1}\sup_{0\leq s\leq t}\sup_{0\leq x\leq 2\pi}|u^{(\epsilon)}(s,x)| \leq \tilde{C}(1,t),$$

where $\tilde{C}(b,t)$ is given by equation (43).

By lemma 12, for each $p > 0$, there is an increasing non negative function $K(p,.)$ such that $K(p,t) < +\infty$ for $t < +\infty$ and $E\{\tilde{C}^p(1,t)\} \leq K(p,t)$. It follows that $\mathbf{Q}$ - almost surely, $u^{(\epsilon)}$ converges weakly in $L^2([0,T]\times[0,2\pi])$ to

$$u(t,x) = \frac{1}{2\pi T}\sum_{mn} -ine^{-(itm\frac{2\pi}{T} + inx)}\lambda_T(m,n).$$

It is standard that **Q** - almost surely, the weak limits of the solutions of equations (81) and (82) solve equations (83) and (84) respectively. On $[0,T]\times[0,2\pi]$, let

$$u^{(\epsilon)}(t,x)=\frac{1}{2\pi T}\sum_{nm}-ine^{(itm\frac{2\pi}{T}+inx)}\lambda_T^{(\epsilon)}(m,n).$$

Then

$$u^{(\epsilon)2}(t,x)=-\frac{1}{(2\pi T)}\sum_{mn}\sum_{m_1n_1}n_1(n-n_1)e^{(itm\frac{2\pi}{T}+inx)}\lambda_T^{(\epsilon)}(m_1,n_1)\lambda_T^{(\epsilon)}(m-m_1,n-n_1)$$

and

$$\begin{aligned}
&\left|\sum_{n_1m_1}n_1(n-n_1)\left(\lambda_T^{(\epsilon)}(m_1,n_1)\lambda_T^{(\epsilon)}(m-m_1,n-n_1)-\lambda_T(m_1,n_1)\lambda_T(m-m_1,n-n_1)\right)\right|\\
&\quad\leq\sum_{m_1n_1}\left|n_1\lambda_T^{(\epsilon)}(m_1,n_1)\right|\left|(n-n_1)\lambda_T^{(\epsilon)}(m-m_1,n-n_1)-(n-n_1)\lambda_T(m-m_1,n-n_1)\right|\\
&\qquad+\sum_{m_1n_1}|(n-n_1)\lambda_T(m-m_1,n-n_1)|\left|n_1\lambda_T^{(\epsilon)}(m_1,n_1)-n_1\lambda_T(m_1,n_1)\right|\\
&\quad\leq\left(\left(\sum_{m,n}n^2|\lambda_T^{(\epsilon)}(m,n)|^2\right)^{1/2}+\left(\sum_{m,n}n^2|\lambda_T(m,n)|^2\right)^{1/2}\right)\\
&\qquad\qquad\times\left(\sum_{mn}n^2\left|\lambda_T^{(\epsilon)}(m,n)-\lambda_T(m,n)\right|^2\right)^{1/2}.
\end{aligned}$$

Firstly,

$$\sum_{m,n}n^2|\lambda_T^{(\epsilon)}(m,n)|^2=\frac{1}{2\pi T}\int_0^T\int_0^{2\pi}|u^{(\epsilon)}(t,x)|^2dxdt<\tilde{C}^2(T)\qquad\forall 0\leq\epsilon<1$$

where $\tilde{C}(T)=\sup_{0\leq\epsilon<1}\sup_{0\leq t\leq T}\sup_{0\leq x\leq 2\pi}|u^{(\epsilon)}(t,x)|$ and $E_{\mathbf{Q}}\left\{\tilde{C}(T)^p\right\}<+\infty$ for all $0<p<+\infty$. This implies that the dominated convergence theorem may be used on

$$\sum_{mn}n^2\left|\lambda_T^{(\epsilon)}(m,n)-\lambda_T(m,n)\right|^2.$$

Since $|\lambda_T^{(\epsilon)}(m,n)-\lambda_t(m,n)|\overset{\epsilon\rightarrow 0}{\longrightarrow}0$ for eacn (m,n), it follows that $u^{(\epsilon)2}$ converges to u^2 and hence that weak limits of the solutions of equations (81) and (82) solve equations (83) and (84) respectively. By integrating equations (81) and (83), it follows that

$$\int_0^T\int_0^{2\pi}u^{(\epsilon)2}(t,x)dxdt\overset{\epsilon\rightarrow 0}{\longrightarrow}\int_0^T\int_0^{2\pi}u^2(t,x)dxdt.$$

Furthermore, since $\mathbf{Q}$ - almost surely, u is the weak limit of $u^{(\epsilon)}$ and $u \in L^2([0,T] \times [0,2\pi])$, it follows that

$$\int_0^T \int_0^{2\pi} u^{(\epsilon)}(t,x)u(t,x)dxdt \stackrel{\epsilon \to 0}{\longrightarrow} \int_0^T \int_0^{2\pi} u^2(t,x)dxdt.$$

It follows that, $\mathbf{Q}$-almost surely, for all $T < +\infty$,

$$\begin{aligned} \int_0^T \int_0^{2\pi} \left(u^{(\epsilon)}(t,x) - u(t,x)\right)^2 dxdt &= \int_0^T \int_0^{2\pi} u^{(\epsilon)2}(t,x)dxdt + \int_0^T \int_0^{2\pi} u^2(t,x)dxdt \\ &\quad -2\int_0^T \int_0^{2\pi} u^{(\epsilon)}(t,x)u(t,x)dxdt \stackrel{\epsilon \to 0}{\longrightarrow} 0. \end{aligned}$$

It follows that $\mathbf{Q}$ - almost surely, $u^{(\epsilon)}(t,x) \stackrel{\epsilon \to 0}{\longrightarrow} u(t,x)$ for Lebesque almost all $(t,x) \in [0,T] \times [0,2\pi]$ for all $T < +\infty$. The dominated convergence theorem therefore gives

$$\begin{aligned} \lim_{\epsilon \to 0} E_{\mathbf{Q}} \left\{ \frac{1}{2\pi} \int_0^T \int_0^{2\pi} u^{(\epsilon)2p}(t,x)dxdt \right\} &= E_{\mathbf{Q}} \left\{ \frac{1}{2\pi} \int_0^T \int_0^{2\pi} \lim_{\epsilon \to +\infty} u^{(\epsilon)2p}(t,x)dxdt \right\} \\ &= E_{\mathbf{Q}} \left\{ \frac{1}{2\pi} \int_0^T \int_0^{2\pi} u^{2p}(t,x)dxdt \right\}, \end{aligned}$$

From the bounds on $u^{(\epsilon)}(t,x)$ uniform in $(\epsilon,t,x) \in [0,1] \times [0,T] \times [0,2\pi]$, convergence of $u^{(\epsilon)}$ to u $\mathbf{Q}$ almost surely for Lebesgue almost all $(t,x) \in [0,T] \times [0,2\pi]$ and convergence of the L^p norms, it follows that $u^{(\epsilon)}$ converges to u in the L^p *norm* topology for each $1 < p < +\infty$. Lemma 22 is proved. □

Proof of theorem 2 This follows directly from lemma 22. □

4 The Invariant Measure for the Stochastic Burgers Equation

The result in this article given by equation (4), concerning the growth of the moments for equation (1) is of interest, following the results found in the article [1]. These results show existence of an *invariant measure* for the viscosity solution of the inviscid Burgers equation

$$\partial_t u + \frac{1}{2}(u^2)_x dt = \partial_t \zeta_x,$$

where the hypotheses on ζ in that article include the hypotheses stated in hypothesis 1 of this article. The viscosity solution is the solution obtained by letting $\epsilon \to 0$ in equation (1). All moments of the invariant measure considered in that article exist, as outlined below. The argument presented by E, Khanin, Mazel and Sinai in [1] is based on Varadhan's theorem from large deviations. Starting from equation (13), solutions to equation (1) are given by the Cole Hopf transformation, equation (28). Consider the *action functional*

$$\mathcal{A}(\xi;0,t)=\frac{1}{2}\int_0^t\dot{\xi}^2(s)ds+\sum_{n=1}^{\infty}a_n\left(\int_0^t\cos(n\xi(s))d_s\beta^{1n}(s)+\int_0^t\sin(n\xi(s))d_s\beta^{2n}(s)\right). \tag{90}$$

It is a relatively straight forward application of Varadhan's theorem from Large Deviation theory to show that

$$\lim_{\epsilon\to 0}-\epsilon\log U^{(\epsilon)}(t,x)=\inf_{\xi:\xi(t)=x}\mathcal{A}(\xi;0,t).$$

It is relatively standard, and is shown in the article [1], that there is a trajectory $(\eta^{(t,x)}(s))_{s\in[0,t]}$ that minimises $\mathcal{A}(.;0,t)$ subject to the constraint that $\eta^{(t,x)}(t)=x$ and that, furthermore, this minimiser satisfies

$$\frac{\partial\eta^{(t,x)}}{\partial s}(s)=-\sum_{n=1}^{\infty}na_n\left(\int_0^s\sin(n\eta^{(t,x)}(r))d_r\beta^{1n}(r)-\int_0^s\cos(n\eta^{(t,x)}(r))d_r\beta^{2n}(r)\right). \tag{91}$$

Set

$$u(t,x)=\frac{\partial}{\partial x}\mathcal{A}(\eta^{(t,x)}(s);0,t).$$

By taking the derivative, integrating by parts and using equation (91), it follows that

$$u(t,x)=\left.\frac{\partial\eta^{(t,x)}(s)}{\partial s}\right|_{s=t}.$$

By lemma 22, u is the limit in L^p norm of $u^{(\epsilon)}$, for any $p<+\infty$. Therefore u satisfies

$$\begin{cases}\partial_t u+\frac{1}{2}(u^2)_x dt=\partial_t\zeta_x\\ u_0\equiv 0.\end{cases} \tag{92}$$

Using this, an upper bound is given in the article [1] for $\sup_{x\in[0,2\pi]}|u(t,x)|$ and the distribution of this upper bound is shown to be independent of t. It is also shown in [1] that all the moments of this distribution exist. An outline of the proof is reproduced here.

Theorem 23. *Let u denote the solution to equation (92). There exist constants $C(p)<+\infty$, independent of t, such that for all $t\in\mathbf{R}_+$,*

$$E_{\mathbf{Q}}\left\{\left(\sup_{0\leq x\leq 2\pi}|u(t,x)|\right)^{2p}\right\}\leq C(p).$$

Proof The analysis follows that given in [1]. It is assumed, following the discussion in [1], that the representation $u(t,x)=\frac{\partial\eta^{(t,x)}}{\partial s}(s)|_{s=t}$ holds, where u is the solution to equation (92), η minimises the action functional (90) and satisfies equation (91). Suppressing the superscripts, let $0\leq t_1\leq t_2\leq t$, then

$$
\begin{aligned}
\dot\eta(t_2)-\dot\eta(t_1) &= -\sum_{n=1}^{\infty} n a_n \int_{t_1}^{t_2} (\sin(n\eta(s)) d_s\beta^{1n}(s) - \cos(n\eta(s)) d_s\beta^{2n}(s)) \\
&= -\sum_{n=1}^{\infty} n a_n ((\beta^{1n}(t_2)-\beta^{1n}(t_1))\sin(n\eta(t_2)) - (\beta^{2n}_{t_2}-\beta^{2n}_{t_1})\cos(n\eta(t_2))) \\
&\quad + \sum_{n=1}^{\infty} n^2 a_n \int_{t_1}^{t_2} \dot\eta(s)(\cos(n\eta(s))(\beta^{1n}(s)-\beta^{1n}(t_1)) + \sin(n\eta(s))(\beta^{2n}(s)-\beta^{2n}(t_1)))ds,
\end{aligned}
$$

so that, setting

$$
C(s,t) = \sum_{n=1}^{\infty} n^2 |a_n| \left(\sup_{s\le r_1\le r_2\le t} |\beta^{1n}(r_2)-\beta^{1n}(r_1)| + \sup_{s\le r_1\le r_2\le t} |\beta^{2n}(r_2)-\beta^{2n}(r_1)| \right),
$$

it follows that

$$
||\dot\eta|(t_2) - |\dot\eta|(t_1)| \le C(t_1,t_2) + \int_{t_1}^{t_2} |\dot\eta|(s) C(t_1,s)ds.
$$

From this, it is straightforward to see that for $s\in[t-1,t]$,

$$
\inf_{t-1\le s\le t} |\dot\eta|(s) \ge |\dot\eta|(t) e^{-C(t-1,t)} - C(t-1,t) \tag{93}
$$

and

$$
\sup_{t-1\le s\le t} |\dot\eta|(s) \le (|\dot\eta|(t) + C(t-1,t)) e^{C(t-1,t)}. \tag{94}
$$

Now, the minimising trajectory minimises the action functional

$$
\begin{aligned}
\mathcal{A}(0,t;\xi) &= \frac{1}{2}\int_0^t \dot\xi^2(s)ds + \sum_{n\ge1} a_n \int_0^t (\cos(n\xi(s))d_s\beta^{1n}(s) + \sin(n\xi(s))d_s\beta^{2n}(s)) \\
&= \mathcal{A}(0,t-1;\xi) + \frac{1}{2}\int_{t-1}^t \dot\xi^2(s)ds \\
&\quad + \sum_{n=1}^{\infty} a_n((\beta^{1n}(t)-\beta^{1n}(t-1))\cos(n\xi(t)) + (\beta^{2n}(t)-\beta^{2n}(t-1))\sin(n\xi(t))) \\
&\quad + \sum_{n=1}^{\infty} n a_n \int_{t-1}^t \dot\xi(s)\left((\beta^{1n}(s)-\beta^{1n}(t-1))\sin(n\xi(s)) - (\beta^{2n}(s)-\beta^{2n}(t-1))\cos(n\xi(s))\right) ds
\end{aligned}
$$

subject to the constraint that $\xi(t)=x$. Setting $\sup_{0\le x\le 2\pi}|u(t,x)| = K$ and $C=C(t-1,t)$ and using the inequalities (93) and (94),

$$
\mathcal{A}(0,t;\xi) - \mathcal{A}(0,t-1;\xi) \ge \frac{1}{2}((|\dot\xi(t)|^2 e^{-C} - C)\vee 0)^2 - C - C||\dot\xi(t)|e^{-C} - C|.
$$

To get an upper bound on $\mathcal{A}(0,t;\xi)-\mathcal{A}(0,t-1;\xi)$, where ξ is the minimiser of $\mathcal{A}(0,t;.)$ subject to $\xi(t)=x$, note that $u(t,x)$ is 2π periodic in x and consider the constant velocity trajectory η such that $\eta(t)=x$ and $\eta(t-1)=\xi(t-1)$ to obtain

$$\mathcal{A}(0,t;\xi)-\mathcal{A}(0,t-1;\xi)\leq 2\pi^2+C+2\pi C.$$

Since this holds for all $x\in[0,2\pi)$, it follows that

$$\frac{1}{2}((Ke^{-C}-C)\vee 0)^2-C-C|Ke^{-C}-C|\leq 2\pi^2+C+2\pi C.$$

It follows that either $K<Ce^{C}$, or

$$\frac{1}{2}K^2e^{-2C}-2KCe^{-C}+\frac{3}{2}C^2-C\leq 2\pi^2+C(1+2\pi),$$

so that

$$(Ke^{-C})^2-4C(Ke^{-C})+(3C^2-4(1+\pi)C-4\pi^2)<0$$

giving

$$\sup_{0\leq x\leq 2\pi}|u(t,x)|\leq K<2Ce^{C}+e^{C}\sqrt{C^2+4(1+\pi)C+4\pi^2}\leq(3C+2(1+\pi))e^{C}<10e^{2C}.$$

To obtain estimates on $E_{\mathbf{Q}}\{K^p\}$, first note that

$$\sup_{t-1\leq r\leq s\leq t}|\beta^{1n}(s)-\beta^{1n}(r)|\leq 2\sup_{t-1\leq s\leq t}|\beta^{1n}(s)-\beta^{1n}(t-1)|,$$

so that, setting $S^{jn}=\sup_{t-1\leq s\leq t}|\beta^{jn}(s)-\beta^{jn}(t-1)|$,

$$E_{\mathbf{Q}}\{|K|^p\}\leq 10^pE_{\mathbf{Q}}\left\{\exp\{4p\sum_{n=1}^{\infty}n^2|a_n|(S^{1n}+S^{2n})\}\right\}=10^p\prod_{n=1}^{\infty}E_{\mathbf{Q}}\left\{\exp\{4pn^2|a_n|S^{1n}\}\right\}^2.$$

An application of lemma 3 gives

$$E_{\mathbf{Q}}\left\{|K|^p\right\}\leq 10^p\exp\left\{32p^2\sum_{n\geq 1}n^4a_n^2+8p(\sqrt{2\log 2}+2\sqrt{2\pi})\sum_{n\geq 1}n^2|a_n|\right\},$$

concluding the proof of theorem 23. This is the line of the proof found in [1]. □

5 Conclusion

There is a striking dichotomy here. The steps that were sketched in section 4 are justified elsewhere in the literature. This is discussed in [1]. The first of these is the application of Varadhan's theorem in the stochastic case. This is straightforward; an integration by parts of the potential term removes the stochastic integral. Secondly, the fact that the action functional, in the stochastic case, has a minimiser and the fact that this minimiser, in the stochastic case, satisfies the Euler - Lagrange equations. These steps are relatively straight forward; after the 'stochastic' integral has been removed by an integration by parts, the proof depends on the $\dot{\xi}^2$ term and the fact that $|\dot{\xi}|$ is raised to a power strictly greater than 1, using standard arguments that date back to Tonelli. Once these steps are justified, the conclusion is that the Choice Axiom leads to inconsistent results. The results from classical dynamics, stating that a minimising trajectory for the action functional exists use crucially the relative weak compactness of the unit ball in L^2. By Tychonoff's theorem, the Choice Axiom implies relative weak compactness of the unit ball in L^2. Kelley [2] showed that relative weak compactness of the unit ball in L^2 implied the Choice Axiom. It is the Choice Axiom, at the countable level, which is employed in the arguments in this article. The conclusion is therefore that this article has provided an example that demonstrates that employing the Choice Axiom leads to contradictory results in analysis.

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Literature Cited

[1] Weinan E, K.Khanin, Mazel, Ya.G. Sinai [2000] *Invariant Measure for Burgers Equation with Random Forcing* Annals of Mathematics, vol. 151 , pp 877 - 960

[2] J.L. Kelley [1950]*The Tychonoff Product Theorem Implies the Axiom of Choice* Fund. Math. 37 pp 75-76

[3] H. Kunita [1990]*Stochastic Flows and Stochastic Differential Equations* Cambridge University Press

[4] D. Revuz and M. Yor [1999] *Continuous Martingales and Brownian Motion* (3rd edition) Springer